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"How did the serpent of inconsistency enter Frege's paradise?"[1]

Crispin Wright

So asks Michael Dummett at the start of chapter 17 of *Frege: Philosophy of Mathematics*.[2] And in the final chapter he suggests an answer: that Frege's major oversight—the key to the collapse of the project of *Grundgesetze*—consisted in

> . . . his supposing there to be a totality containing the extension of every concept defined over it; more generally [the mistake] lay in his not having the glimmering of a suspicion of the existence of indefinitely extensible concepts.[3]

The diagnosis is repeated in the essay, "What is Mathematics About?", where Dummett writes that

> Frege's mistake . . . lay in failing to perceive the notion [of a value-range] to be an indefinitely extensible one, or, more generally, in failing to allow for indefinitely extensible concepts at all.[4]

Now, claims of the form,

> Frege fell into paradox because…….

[1] This article draws on and develops material originally presented at the *Fregefest* at the University of California, Irvine in February 2006 that later provided for my lecture, "Whence the Paradox? Axiom V and Indefinite Extensibility", given at the 4th International Lauener Symposium on Analytical Philosophy in Bern in May 2010. The Symposium was dedicated to the work of the late Sir Michael Dummett, who attended to receive the Lauener Prize for an Outstanding Oeuvre in Analytical Philosophy on the same occasion. A transcript of my lecture is published in W. Essler and M. Frauchiger, eds. (2015)

More recent presentations of the main ideas were at the Dummett symposium in Leeds in September 2013, and at the *Grundgesetze* workshop in New York, the Oslo workshop on *Abstraction*, and the *Metaphysical Basis of Logic* seminar at the Northern Institute of Philosophy in Aberdeen, all in May of 2014. Thanks to the participants on all these occasions for very helpful feedback. Special thanks to Richard Heck, Øystein Linnebo and Stewart Shapiro for generous and helpful comments on an earlier draft which have saved me from a number of confusions, and especially to Stewart for the shared research that provides the platform of the paper, and for countless conversations about the issues over the years.

[2] Dummett (1991)

[3] Dummett (1991), 317

[4] Dummett (1993), p. 441

are notoriously difficult to assess even when what replaces the dots is relatively straightforward. Paradoxes of any depth are usually complex and seldom involve moves that, once exposed, allow of straightforward identification as clear-cut "mistakes". The paradox attending Law V is no exception. Diagnostic offerings have included —

(A) *Unrestricted quantification*: Frege fell into paradox because he allowed himself to quantify over a single, all-inclusive domain of objects (Russell, Dummett).

(B) *Impredicative objectual quantification*: Frege fell into paradox because he allowed himself to define value-ranges using (first order) quantifiers ranging unrestrictedly over those very objects (Russell, Dummett).

(C) *Impredicative higher-order quantification*: Frege fell into paradox because he allowed himself to formulate conditions on value-ranges using (higher-order) quantifiers ranging over those very conditions (Russell, Dummett).

(D) *Inflation*: Frege fell into paradox because he adopted an axiom— Law V—which is *inflationary*, i.e. defines its proper objects by reference to an equivalence on concepts that partitions the higher-order domain into too many cells (Boolos, Fine).

And while it is indeed clear that Frege did do all these things, — and prior to that, clear, or anyway relatively clear, what it *is* to do them, — the diagnoses presented are all nevertheless problematic. Contra (A), for example, there are multiple instances where unrestricted (objectual) quantification seems both intelligible and essential to the expression of the full range of our thoughts. Contra (B) and (C), while impredicative quantification of both first and higher-orders is indeed essential to the generation of the paradox, it is also essential to a range of foundational moves in classical mathematics and, in so far as it may seem objectionable, the objections seem more properly epistemological than logical. Contra (D), there is no straightforward connection, in a higher-order setting, between unsatisfiability and inconsistency; and it is salient in any case that the actual derivation of the contradiction from Frege's axiom nowhere implicitly depends upon an assumption of the classical range of the higher-order variables but would go through on, for example, a substitutional interpretation of second-order quantification. However with Dummett's proposal cited above:

(E) Frege fell into paradox because he didn't have even a glimmering of a suspicion of the existence of indefinitely extensible concepts,

matters may seem yet worse. This diagnosis may seem not to get so far as proposing *any* definite account of Frege's "colossal blunder" (as Dummett elsewhere characterises it[5]) at all, even a controversial one. What exactly did Frege do, or fail to do, because he failed to reckon with the indefinite extensibility of *extension* or *value-range*? What indeed exactly *is* indefinite extensibility? The notion continues to be met with the kind of scepticism which George Boolos espoused when he roundly rejected Dummett's diagnosis, opining that it was "*To his credit*, [that] Frege did not have the glimmering of a suspicion of the existence of indefinitely extensible concepts."[6] [My emphasis.]

Indefinite extensibility has been connected in recent philosophy of mathematics with many large issues, including not just the proper diagnosis of the paradoxes, but the legitimacy of unrestricted quantification, the content of quantification (if legitimate at all) over certain kinds of large populations, the legitimacy of classical logic for such quantifiers, the proper conception of the infinite, and the possibilities for (neo-) logicist foundations for set theory.[7] But my project here is limited to the appraisal of Dummett's diagnosis. I shall address a problem that obscures the usual intuitive characterisations of the notion of indefinite extensibility, and offer thereby what I believe to be the correct characterisation of the notion. En passant, we shall review some issues about the "size" of indefinitely extensible concepts. And that will bring us into position to scrutinise the connections of the notion as characterised with paradox, and specifically the paradox that Russell found for Law V. A full enough plate.

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[5] Dummett (1994), p. 243

[6] Boolos (1998), at p. 224. I should observe, though, that, in context, Boolos is assuming that an indefinitely extensible concept comes with a prohibition on unrestricted quantification over its instances—something that Dummett repudiates in his response.

[7] Some of these issues are further pursued in Shapiro and Wright (2006). The first part of my discussion here will draw extensively on aspects of that paper.

1. Naïveté or Insouciance?

I should begin, though, with a short digression and disclaimer. When I speak of "Dummett's diagnosis", I intend no implication of uniqueness nor to take sides on an issue which, I believe, should cause interpreters of Dummett some head-scratching. As noted above, Dummett has made a number of not obviously equivalent observations about the genesis of the paradox. My own sense — I will not try to substantiate it here — is that over the years between the publications of *Frege: Philosophy of Language* and *Frege: Philosophy of Mathematics* he changed his mind about unrestricted objectual quantification, at first regarding it as illegitimate but later coming to allow that there are legitimate such generalisations but that they must be understood non-classically (truth-conditionally). But change of mind is not, presumably, at work within the pages of *Frege: Philosophy of Mathematics*. Yet in the chapter whose opening line provides our title, there is no mention of indefinite extensibility. Suspicion is cast, rather, on second-order quantification, of which Dummett writes that "it is to its presence in Frege's formal language that the contradiction is due",[8] alleging a little later that it was Frege's "amazing insouciance concerning the second order quantifier"[9] that was the primary reason for his falling into inconsistency. The diagnosis that places the blame on Frege's innocence of the notion of indefinite extensibility occurs seven chapters later. These do not seem to be the same diagnosis. What is going on?

Dummett's charge of insouciance refers to the extremely and uncharacteristically sketchy remarks that Frege offers in *Grundgesetze* by way of explanation of second order generality.[10] But the main burden of the latter part of ch. 17 of *Frege: Philosophy of Mathematics* is to explain in some detail what goes wrong with Frege's purported proof of consistency (that every name formed in the language of his formal system has a reference) at *Grundgesetze* §31. What Frege

[8] Dummett (1991), p. 217

[9] Dummett (1991), p. 218

[10] At Grundgesetze §20 and §25. See Dummett (1991) at pp. 217-8

needs to show for the case of second-order quantification is that any sentence (for Frege, name) formed by second order quantification into a first-order sentence (name) that has a reference likewise has a reference. As Dummett very clearly explains, the attempt founders on the circularity engendered by the fact that the relevant first-order sentences may themselves contain occurrences of Frege's second-order quantifier. But this merely deepens the exegetical puzzle. Perhaps Frege's carelessness in characterising the second-order quantifier contributed to his overlooking the shortcomings of the argument of *Grundgesetze* §31. But how do the shortcomings of a purported consistency proof contribute to the explanation of the inconsistency of the system it concerns?[11]

The apparent tension is observed by Boolos who archly remarks that "One might wonder whether it was Frege's insouciance or his naïveté that Dummett thinks is to blame for the error."[12] However Boolos goes on to say that he thinks the Dummett does have a unitary account of the source of the contradiction to offer, and then draws attention to two points on Dummett's behalf. The first is that Law V is consistent in a first-order setting.[13] The second is that, for any domain of sets, D, the Russell paradox shows—"according to Dummett"—that no set containing exactly the elements of D that are not self-membered can also be an element of D: the Russellian functor, "The set of elements of x that are not members of themselves," where x is some domain of sets, always forces an extension of the domain. But, Boolos observes, the expression of that functor in Frege's formal language will require second-order quantification; thus

> . . . the introduction of second order quantifiers forces an extension of the domain to comprise such new objects . . . Once second-order quantifiers are added, no domain is large enough to contain all extensions of concepts defined on that domain. It was because Frege didn't have a glimmering of a suspicion of the way each domain must give rise to a properly wider one that he could be insouciant about the second-order quantifier."[14]

[11] Compare "A failed consistency proof for a consistent system would hardly be a serpent in Eden", Boolos (1998) p. 224

[12] Boolos (1998) p. 220

[13] Something first proved, I believe, by Terence Parsons (1987)

[14] Boolos, op.cit.p. 221

Well, other readers may do better, but I find it hard to get any clear sense of what exactly is the "unitary" account that Boolos intends these remarks to convey. Russell's paradox does indeed depend on the naïve assumption that the Russellian function on a domain D takes a value within D— on ignoring its expansionist tendency, as it were; and the definition of the function does indeed require second-order resources. But, notwithstanding the inadequacy of Frege's explanation of the latter and his, perhaps consequential, reliance upon a flawed proof that all was safe, the question is: are those resources in good standing or are they not? If they are, they are in no way to blame for the paradox. If they are not, the Russellian function is ill-defined and thus has no tendency to show that "each domain must give rise to a properly wider one".

It might be suggested that all Dummett need be taken to intend is that Frege's casual handling of second-order quantification in the setup of the system of *Grundgesetze* resulted in his being less than circumspect about the range of resources it provides for the definition of first-order functions, and that his innocence of the notion of indefinite extensibility will have prevented his realising the pressing need for such circumspection. Well, maybe. And Boolos' words will perhaps bear that interpretation. But notice that that is to change the question. It is to offer an explanation of why Frege didn't realise that he had left the door open, as it were. But it is clear that what Dummett means to be addressing is not that question, but the question why the door was open in the first place. We are looking for a diagnosis of the source of the contradiction, not of Frege's oversight of it

In his Chairman's Address[15], Dummett responds directly to Boolos' remarks. A crucial passage runs as follows:

> The context principle required the reference of the terms of the theory to be stipulated by laying down the values of functions, including concepts, taking their referents as arguments, the whole procedure being validated by a proof that a unique reference had been stipulated for every well-formed expression: Frege's consistency proof.
>
> For Frege, and for anyone who believes his justification for speaking of abstract objects to be in part correct, the problem is not so much what made his theory inconsistent as how, in the face of the semantics he devised for it, it *could* have been inconsistent. Boolos remarks that a failed

[15] Dummett (1994)

> consistency proof for a consistent system would hardly be a serpent in Eden. This dismissive observation would be just if the proof were a mere bright idea appended by Frege to his main exposition. It was not: it was integral to his entire conception of the manner in which to justify introducing a range of abstract objects.
>
> Second-order quantification was essential for the inconsistency . . .

I would want to resist the underlying train of thought here. It is perfectly possible to accept that reference may be conferred upon a class of abstract singular terms in the kind of way that Frege proposed— roughly, by stipulating the content of complex expressions in which they occur in such a way that suitable (atomic) such expressions (sentences) have reference (are true)— without any liability to paradox, even when second-order resources are freely deployed in the stipulations. So much, anyway, is the intended lesson of the modern neo-Fregean constructions of arithmetic and analysis. Still, there is an apparent lacuna between the contention, correct or not, that Frege's method of introducing abstract singular terms was essentially put in disorder by its ungroundedness/circularity and the diagnosis of the paradox that Dummett seems here to be suggesting. It is as if, in Dummett's view, the contradiction is merely a dramatic, occasional symptom of this underlying disorder— one that emerges in the environment of (Frege's casual handling of) the second-order quantifier—but the disorder is there anyway, rather as a lip blister can emerge, when one has a cold, as a symptom of underlying infection with the herpes virus. The reader may find this idea less problematic than I (and Boolos.)

The point remains, though, that this train of thought of Dummett's, whatever insight it may prove to contain, seems to have little to do with indefinite extensibility. We are no closer to a "unitary account" of Dummett's thinking on the issue and I shall not here attempt to explore further whether such an account is possible. What is unquestionable is that there are intricate and important questions about the role of second-order logic in the paradox, which we will come to later.

2. Indefinite Extensibility: the problem of characterisation

The suggestion that indefinite extensibility is playing some kind of devil's part in the paradoxes is of course anticipated in Russell, whose [1906] concludes:

> . . . the contradictions result from the fact that . . . there are what we may call *self-reproductive* processes and classes. That is, there are some properties such that, given any class of terms all having such a property, we can always define a new term also having the property in question. Hence we can never collect *all* of the terms having the said property into a whole; because, whenever we hope we have them all, the collection which we have immediately proceeds to generate a new term also having the said property.

Compare Dummett: an

> *indefinitely extensible* concept is one such that, if we can form a definite conception of a totality all of whose members fall under the concept, we can, by reference to that totality, characterize a larger [sic]totality all of whose members fall under it.[16]

According to Dummett, an indefinitely extensible concept *P* has a "principle of extension" that takes any definite totality *t* of objects each of which has *P*, and produces an object that also has *P*, but is not in *t*.[17] But what does "definite" mean in that? Presumably a concept *P* is *definite* for Dummett's purpose in those passages just if it is not indefinitely extensible! If so, then Dummett's remarks won't do as a definition, even a loose one, since they appeal to its complementary "definite" to characterize what it is for a concept to be indefinitely extensible. And Russell, of course, does no better by speaking unqualifiedly of "any class of terms all having such a property", since he is taking it for granted that classes, properly so regarded, are "wholes" or "have a total"—that is, presumably, are definite in the relevant complementary sense.

Notice that it would not do just to drop any reference to definiteness, or an equivalent, in the intuitive characterisation. If the suggestion had been, for example, that an

> *indefinitely extensible* concept is one such that, for any given totality all of whose members fall under the concept, we can, by reference to that totality, characterize a larger totality all of whose members fall under it,

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[16] Dummett (1993), 441

[17] See also Dummett (1991), 316-319, where he cites the above passage from Russell.

then the usual suspects would fail the test— if we took *set*, for instance, as the target concept and then picked as the first mentioned "totality" simply the sets themselves, there would be no "larger" totality of sets to extend into. And if we then stipulated instead that attention should be restricted to *proper* sub-totalities, then all concepts would, trivially, pass the test.

This problem of implicit circularity in the intuitive characterization of indefinite extensibility is a serious one. Indeed, it is the major difficulty in forming a clear idea of the notion. But it would be premature to lose confidence in the notion of indefinite extensibility because of it. A reminder may be helpful how the three concepts targeted by the classic set-theoretic paradoxes — Burali, Cantor, and Russell — do seem to present a suggestive common pattern:

(1) *Ordinal*. Think of the ordinals in an intuitive way, simply as order-types of well-orderings. Let O be any definite collection of ordinals. Let O' be the collection of all ordinals smaller than or equal to some member of O. O' is well-ordered under the natural ordering of ordinals, so has an order-type — γ. So γ is itself an ordinal. Let γ' be the order-type of the well-ordering obtained from O' by tacking an element on at the end. Then γ' is an ordinal number, and γ' is not a member of O. So *ordinal number* is indefinitely extensible.[18]

(2) *Cardinal*. Let C be any definite collection of cardinal numbers. Assign to each of its members a set of that exact cardinality, and form the union of these sets, C'. By Cantor's theorem, the collection of subsets of C' is larger than C', so larger than any cardinal in C. So *cardinal number* is indefinitely extensible.

(3) *Set/class*. Dummett writes

> "Russell's concept *class not a member of itself* provides a beautiful example of an indefinitely extensible concept. Suppose that we have conceived of a class C all of whose members fall under the concept. Then it would certainly involve a contradiction to suppose C to be a member of itself. Hence, by considering the totality of the members of C together with C itself, we have specified a more inclusive totality than C all of whose members fall under the concept *class not a member of itself*."[19]

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[18] As Dummett puts it,

> . . if we have a clear grasp of any totality of ordinals, we thereby have a conception of what is intuitively an ordinal number greater than any member of that totality. Any definite totality of ordinals must therefore be so circumscribed as to forswear comprehensiveness, renouncing any claim to cover all that we might intuitively recognise as being an ordinal. ((1991) at 316.)

[19] Dummett (1993) at p. 441

> Observe that it follows that *set* itself is indefinitely extensible, since any definite collection—set—of sets must omit the set of all of its members that do not contain themselves.

To be sure, some of the argumentation involved in these cases is potentially contestable. Someone could challenge the various set-theoretic principles (Union, Replacement, Power-set, etc.) that are implicitly invoked, for instance. But it seems reasonable to agree with Russell and Dummett that the examples do exhibit, prima facie, some kind of "self-reproductive" feature. The question is whether we can give a more exact, philosophically robust and useful characterisation of it.

3 **Indefinite extensibility and the ordinals: Russell's Conjecture and 'small' cases.**

We can make a start by following up on a suggestion of Russell himself. Russell writes that it "is probable" that if *P* is any concept which demonstrably "does not have an extension", then "we can actually construct a series, ordinally similar to the series of all ordinals, composed entirely of terms having the concept *P*". The conjecture is in effect that if *P* is indefinitely extensible, then there is a one-to-one function from the ordinals into *P*.[20]

If Russell is right, then any indefinitely extensible concept determines a collection at least as populous as the ordinals — so, one might think, surpassing populous! And in that case one might worry whether the connection made by Russell's Conjecture is acceptable. For Dummett himself at least has characteristically taken it that both the natural numbers and real numbers are indefinitely extensible totalities in just the same sense that the ordinals and cardinals are, with similar consequences, in his opinion, for the understanding of quantification over them and the standing of classical logic in the investigation of these domains. Moreover in the article which contains his earliest published discussion of the notion,[21] Dummett argues that the proper interpretation of Gödel's incompleteness theorems for arithmetic is precisely to teach that *arithmetical truth* and *arithmetical proof* are also both indefinitely extensible concepts—yet

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[20] Russell (1906), p. 144

[21] Dummett (1963)

neither presumably has an even more than countably infinite extension, still less an ordinals-sized one. (The ordinary, finitely based language of second-order arithmetic presumably suffices for the expression of any arithmetical truth.) It would be disconcerting to lose contact with perhaps the leading modern proponent of the importance of the notion of indefinite extensibility so early in the discussion. But then who is mistaken, Russell in 1906[22] or Dummett? Can there be "small" indefinitely extensible concepts?

The issue will turn out to be important for the proper understanding of indefinite extensibility. To fix ideas, consider the so-called Berry paradox, the paradox of "the smallest natural number not denoted by any expression of English of fewer than 17 words". Here is a statement of it. Define an expression *t* to be *numerically determinate* if *t* denotes a natural number and let *C* be the set—assuming there is one—of all numerically determinate expressions of English. Consider the expression: "The smallest natural number not denoted by any expression in *C* of fewer than 17 words." Assume that this is a numerically determinate expression of English. Then contradiction follows from that assumption, the assumption that the set C exists, and the empirical datum that *b* has 16 words (counting the contained occurrence of '*C*' as one word).

The analogy with the classic paradoxes may look good: a principle of extension seemingly inbuilt into a concept leads to aporia when applied to a totality supposedly embracing all instances of the concept. But, as emerges if we think the process of "indefinite extension" through, there are complications.

To see why, let an initial collection, *D,* consist of just the ten English numerals, "zero" to

[22] It is relevant to recall that Russell (1908) himself, in motivating a uniform diagnosis of the paradoxes, included in his list of chosen examples some at least where the "self-reproductive" process seems bounded by a relatively small cardinal. For instance the Richard paradox concerning the class of decimals that can be defined by means of a finite number of words makes play with a totality which, if indeed indefinitely extensible, is at least no greater than the class of decimals itself, i.e. than $2^{\aleph_0}$. Was Russell simply unaware of this type of example in 1906, when he proposed the conjecture discussed above? Or did he not in 1906 regard the Richard paradox and others involving "small" totalities as genuine examples of the same genre, then revising that opinion two years later?

"nine". Count 'D', so defined, as part of English, and consider "the smallest natural number not denoted by any member of D of fewer than 17 words". Call this 16-worded expression "W_1". Its denotation, clearly, is 10. W_1 is a numerically determinate expression of English, but not in D. Let D_1 be $D \cup \{W_1\}$. Count ' D_1' as an English one-word name. Now repeat the construction on D_1, producing W_2. Let D_2 be $D_1 \cup \{W_2\}$. Count ' D_2' as an English one-word name. Do the construction again. Keep going . . .

How far can you keep going? Well, not into the transfinite. For reflect that 0 to 9 are all denoted by single-word members of D; 10 is denoted by the 16-worded "the smallest natural number not denoted by any member of D of fewer than 17 words"; 11 is denoted by the 16-worded "the smallest natural number not denoted by any member of D_1 of fewer than 17 words"; 12 is denoted by the 16-worded "the smallest natural number not denoted by any member of D_2 of fewer than 17 words"; and so on. So every natural number is denoted by some expression of English of fewer than 17 words. So the "the smallest natural number not denoted by any expression in C of fewer than 17 words" *has no reference*—and hence is not a numerically determinate expression after all, contrary to the assumptions of the paradox.

This result does not immediately give us the last word on the Berry paradox, since it depends on assumptions about English — specifically, that it may be reckoned to contain all the series of names, *D, D1, D2*, etc., and that these can be reckoned to be one-word names — which may be rejected.[23] The point I am making, rather, is that, *when* the relevant assumptions about what counts as English are allowed, the construction shows that while there is indeed a *kind* of

[23] What if we do not make those assumptions? Well, even so, the point stands in general that for any condition, C, on numerically determinate expressions, the template:

> The least natural number not denoted by any C-expression of less than such-and-such a degree of complexity,

cannot always generate an expression which (i) is numerically determinate, (ii) is itself of less than such-and such a degree of complexity and (iii) satisfies C. In the worked example, C—English—was characterised in such a way that (i) proved to fail. If the paradox-monger so characterises C as to deny the resources for that upshot, we can expect that his enterprise will be frustrated by failure of (ii) and/or (iii) instead.

indefinite extensibility about the concept*, numerically determinate expression of English*, it is—if I may be allowed the oxymoron—a *bounded* indefinite extensibility: indefinite extensibility up to a limit; in this case the first transfinite ordinal, ω. When the limit is reached, the result of the construction is a (presumably) *definite* collection of entities that does not in turn admit of extension by the original operation. So the targeted concept will not be *indefinitely* extensible, at least not in the spirit of Dummett's and Russell's intuitive characterisations.

Consider another example. As noted above, Dummett [1963] contends that Gödel's incompleteness theorem shows that *arithmetical truth* is indefinitely extensible. It may seem clear enough what he has in mind. It is straightforward to initiate something that looks like a process of "indefinite extension". Just let A_0 be the theorems of some standard axiomatisation of arithmetic. For each natural number n, let A_{n+1} be the collection A_n together with a Gödel sentence for A_n. Presumably, if A_n is Definite, then so is A_{n+1}, and, of course, A_n and A_{n+1} are distinct. Unlike the case of the Berry paradox, we know that this construction can indeed be continued into the transfinite. Let A_ω be the union of $A_0, A_1, \ldots$ Arguably, A_ω too is definite. Indeed, if A_0 is recursively enumerable, then so is A_ω. Thus, we can obtain $A_{\omega+1}, A_{\omega+2}, \ldots$ and so on. Then we may take the union of those to get $A_{2\omega}$, and onward, "Gödelising" all the way.[24]

On the usual, classical construal of the extent of the ordinals, however, this process too cannot continue without limit, but must "run out" well before the first uncountable ordinal. Let λ be an ordinal and let us assume that we have obtained A_λ. The foregoing construction will take us on to the next set $A_{\lambda+1}$ only if the collection A_λ has a Gödel sentence. And that will be so only if

[24] I don't think it matters for the purposes of the example but I am aware that I am slurring over some complicated mathematical issues here concerning how exactly, where λ is a limit ordinal, one is to arrive at a recursively characterised, so "Gödelisable", specification of A_λ.

A_λ is *recursively axiomatisable*. But clearly it cannot be the case that for every (countable) ordinal λ, A_λ is recursively axiomatisable. For there are uncountably many countable ordinals but only countably many recursive functions.

4 Indefinite extensibility explicated.

Let's take stock. Russell's Conjecture, that indefinitely extensible concepts are marked by the possession of extensions into which the classical ordinals are injectible, still stands. At any rate some apparent exceptions to it, like *numerically determinate expression of English* (when "English" is understood to have the expressive resources deployed above) and *arithmetical truth*, are not really exceptions. For the principles of extension they involve are not truly *indefinitely* extensible but stabilise after some series of iterations isomorphic to a proper initial segment of the ordinals[25]—at least if the ordinals are allowed their full classical extent.

That said, though, the point remains that Russell's Conjecture, even should it be extensionally correct, is certainly not the kind of characterisation of indefinite extensibility we should like to have. If Russell's Conjecture were the best we could do, it would be a *triviality* that the ordinals themselves are indefinitely extensible. What is wanted is a perspective from which we can explain *why* Russell's Conjecture is good, if indeed it is—equivalently, a perspective from which we can characterise exactly what it is about *ordinal* that *makes* it the paradigm of an indefinitely extensible concept.

So let's step back. An indefinitely extensible totality *P* is intuitively unstable, "restless", or "in growth". Whenever you think you have it safely corralled in some well-fenced enclosure, suddenly—hey presto!—another fully *P*-qualified instance pops up outside the fence. The primary problem in clarifying this kind of figure is to dispense with the metaphors of the style of

[25] That's the correct thing to say about the Berry case as constructed. For the arithmetical truth and "Gödelisation" example, the scope for iterated application of the principle of extension peters out before a certain ordinal number of repetitions is reached but has no specific ordinal lowest bound.

"well-fenced enclosure" and "growth". Obviously a claim is intended about sub-totalities of P and functions on them to (new) members of P. But, as we observed, the intended claim does not concern *all* sub-totalities of P: we need to say for *which kind* of sub-totalities of P the claim of extensibility within P is being made. If we could take it for granted that the notion of indefinite extensibility is independently clear and in good standing and picks out a distinctive type of totality, then we could characterize the relevant kind of sub-totality exactly as Dummett did— they are the sub-totalities that are, by contrast, *definite*. For the indefinite extensibility of a totality, if it consists in anything, precisely consists in the fact that any definite sub-totality of it is merely a *proper* sub-totality. But at this point the clarity and good standing of the notion of infinite extensibility may not yet be taken for granted.

Here is the promised way forward. Let us, at least temporarily, finesse the "which sub-totalities?" issue by starting with an explicitly relativised notion. Let P be a concept of items of a certain type τ. Typically, τ will be the (or a) type of individual objects. Let Π be a higher-order concept— a concept of concepts of type τ items. Let us say that P is *indefinitely extensible with respect to* Π if and only if there is a function F from items of the same type as P to items of type τ such that if Q is any sub-concept of P such that ΠQ then

(1) FQ falls under the concept P,
(2) It is not that case that FQ falls under the concept Q, and
(3) $\Pi Q'$, where Q' is the concept instantiated just by FQ and by every item which instantiates Q (i.e., $\forall x(Q'x \equiv (Qx \vee x = FQ))$; (in set-theoretic terms, then, Q' is $(Q \cup \{FQ\})$.

Intuitively, the idea is that the sub-concepts of P of which Π holds have no maximal member. For any sub-concept Q of P such that ΠQ, there is a proper extension Q' of Q such that $\Pi Q'$.

This relativised notion of indefinite extensibility is quite promiscuous, covering a lot of different examples. Here are three:

(*Natural number*): Px iff x is a natural number ; ΠQ iff the Qs (i.e the instances of Q) are finite in number; FQ is the successor of the largest *instance of* Q.

Thus *natural number* is indefinitely extensible with respect to *finite*.

(*Real number*): *Px* iff *x* is a real number; Π*Q* iff the *Q*s are countably infinite. Define *FQ* using a Cantorian diagonal construction.

Thus *real number* is indefinitely extensible with respect to *countable*.

(*Arithmetical truth*): *Px* iff *x* is a truth of arithmetic; Π*Q* iff the *Q*s are recursively enumerable. *FQ* is a Gödel sentence generated by the *Q*s.

Since *FQ* is a truth of arithmetic and is not one of the *Q*s, *arithmetical truth* is indefinitely extensible with respect to *recursively enumerable*.

And naturally the three principal suspects are covered as well:

(*Ordinal number*): *Px* iff *x* is an ordinal ; Π*Q* iff the Qs exemplify a well-ordering type, γ (which since Q is a sub-concept of *ordinal*, they will.) *FQ* is the successor of γ.

Thus *ordinal number* is indefinitely extensible with respect to the property of *exemplifying a well-ordering type*

(*Cardinal number*): *Px* iff *x* is a cardinal number; Π*Q* iff the *Q*s compose a set. *FQ* is the power set of the union of a totality containing exactly one exemplar set of each *Q* cardinal.

Thus *cardinal number* is indefinitely extensible with respect to the property of *composing a set*.

(*Set*): *Px* iff *x* is a set; Π*Q* iff the *Q*s compose a set. *FQ* is the set of *Q*s that are not self-members.

Thus *set* is indefinitely extensible with respect to the property of *composing a set*.

This relativised notion of indefinite extensibility should impress as straightforward enough, but it does not, of course, shed any immediate philosophical light on the paradoxes. Our goal remains to define a notion that may shed such light, an unrelativised notion of indefinite extensibility that still covers *ordinal number*, *cardinal number*, and *set* but somehow illuminates why they are associated with paradox while *natural number*, *real number* and *arithmetical truth* are not. So what next?

Three further steps are needed. Notice to begin with that the listed examples sub-divide into three kinds. There are those where—helping ourselves to the classical ordinals—we can say that some ordinal λ places a lowest limit on the length of the series of Π-preserving applications of *F* to any *Q* such that Π*Q*. Intuitively, while each series of extensions whose length is less than

λ results in a collection of *P*'s which is still Π, once the series of iterations extends as far as λ, the resulting collection of *P*'s is no longer Π, and so the "process" stabilises. This was the situation noted with *numerically determinate expression of English* under the assumptions of our discussion of the Berry paradox, and is also the situation of the first example above. Then there are cases where, whist we may not be able to nominate any particular classical ordinal as a *lowest* limit on the length of the series of Π-preserving applications of *F* to any *Q* such that ΠQ, we can at least identify specific ordinal limits. That is, arguably, the case for the second two listed examples. But neither of these is the situation with the principal suspects: in those cases there is *no* classical ordinal limit to the Π-preserving iterations. With *ordinal number*, this is obvious, since the higher-order property Π in that case just is the property of having a well-ordering type. Indeed, let λ be any ordinal. Then the first λ ordinals have the order type λ and so they have the property. The "process" thus does not terminate or stabilise at λ. With *set* and *cardinal number*, we get the same result if we assume that for each ordinal λ, any totality that has order type λ is a set and (thus) has a cardinality.

Let's accordingly refine the relativised notion to mark this distinction. So first, for any ordinal λ say that *P* is *no-more-than-λ-extensible with respect to* Π just in case *P* and Π meet the conditions for the relativised notion as originally defined but λ places a limit on the length of the series of Π-preserving applications of *F* to any sub-concept *Q* of *P* such that ΠQ. Next, say that *P* is *properly indefinitely extensible* with respect to Π just if *P* meets the conditions for the relativised notion as originally defined and there is no λ such that *P* is no-more-than-λ-extensible with respect to Π. Finally, say that *P* is *indefinitely extensible* (simpliciter) just in case there is a Π such that *P* is properly indefinitely extensible with respect to Π.

My suggestion, then, is that the circularity involved in the apparent need to characterize indefinite extensibility by reference to *definite* sub-concepts/collections of a target concept *P* can be finessed by appealing instead at the same point to the existence of some species—Π—of sub-concepts of *P*/collections of *P*s for which Π-hood is *limitlessly* preserved under iteration of the relevant operation.

This notion still contains, N.B., a relativity. Indefinite extensibility, so characterised, is relative to one's conception of what constitutes a *limitless* series of iterations of a given operation. No doubt we start out innocent of any conception of serial limitlessness save the one implicit in one's first idea of the infinite, whereby any countable potential infinity is limitless. Under the aegis of this conception, *natural number* is properly indefinitely extensible with respect to *finite* and so, just as Dummett suggests, indefinitely extensible simpliciter. The crucial conceptual innovation which transcends this initial conception of limitlessness and takes us to the ordinals as classically conceived is to add to the idea that every ordinal has a successor the principle that every infinite series of ordinals has a limit, a first ordinal lying beyond all its elements—the resource encapsulated in Cantor's Second Number Principle. If it is granted that this idea is at least partially—as it were, initial-segmentally—acceptable, the indefinite extensibility of *natural number* will be an immediate casualty of it. (Critics of Dummett who have not been able to see what he is driving at are presumably merely taking for granted the orthodoxy that the second number principle is at least partially acceptable.)

5 Indefinite extensibility: Burali-Forti

Very well. Roughly summarized, then, the proposal is that *P* is indefinitely extensible just in case, for some Π, any Π sub-concept of *P* allows of a *limitless* series of Π-preserving enlargements. Since the series of Π-preserving enlargements is limitless, any such concept P must indeed allow of an injection of the ordinals into its instances, so Russell' s conjecture is confirmed by this account. It is immediately striking, though, that there seems to be nothing automatically paradoxical about indefinite extensibility, so characterised. Why should a concept in good standing not be sufficiently "expansive" to contain a limitlessly expanding series of Π sub-concepts without ever puncturing, as it were? I'll return to this below.

Still, there is a connection with paradox nearby. For example, in case *P* is *ordinal*, and Π*Q* holds just if the *Q*'s *exemplify a well-order-type*, it seems irresistible to say that *ordinal* is itself Π. After all, the ordinals are well-ordered. But then the relevant principle of extension, *F*, kicks in and dumps a new object on us that both must and cannot be an ordinal—must because it corresponds, it seems, to a determinate order-type; but cannot because the principle of extension always generates a non-instance of the concept to which it is applied. Thus runs the Burali-Forti paradox.

The question, therefore, is why we have allowed our intuitive concept of *ordinal* to fall, fatally, within the compass of the relevant Π/F pair? For *that*, it may seem, is the key *faux pas*.

Well, but what option did we have? There is no room for question whether the ordinals are well ordered. But to be well-ordered is to have an order-type, and we have identified the ordinal numbers with order-types. The only move open, it seems, is to deny that every well-ordered series *is* of a determinate order-type, has an ordinal number, Specifically, it seems we have to deny that *ordinal* itself determines a well-ordered series of a determinate order type and so has an ordinal number. But we need—as too few theorists of these matters have been wont to do—to take the measure of the price of that denial. The price is that before we can assure ourselves of the existence of any particular limit ordinal, we need first to know that its putative predecessors are not 'all the ordinals there are'. And this price will be exacted right back at first base, when the issue is that of justifying the existence of ω, the limit of the finite ordinals. In short, the pressure that induces the *faux pas* is just the pressure to allow the ordinals to run into the Cantorian transfinite in a principled fashion in the first place. The Burali-Forti paradox, and the more general predicament of *ordinal number* that it brings out, thus seems aptly described as indeed exactly a paradox of indefinite extensibility.

Say that P is *reflexively indefinitely extensible* just in case P is indefinitely extensible as characterised above and, in addition, ΠP—i.e. P itself satisfies, or anyway intuitively *ought* to satisfy, the trigger concept Π. Reflexive indefinite extensibility is the notion that has the intrinsic connection with paradox that we have been looking for. Any reflexively indefinitely extensible concept will generate a paradox of the broad structure of the Burali-Forti paradox. *Ordinal*, as intuitively, innocently understood, is reflexively indefinitely extensible.

6 Indefinite extensibility: Cantor

How close is the comparison provided by *cardinal* number and Cantor's paradox? These remarks of Dummett suggest that he regards the situation as a tight parallel:

> . . . to someone who has long been used to finite cardinals, and only to [finite cardinals], it seems obvious that there can only be finite cardinals. A cardinal number, for him, is arrived at by counting; and the very definition of an infinite totality is that it is impossible to count it. . . . [But this] prejudice is one that can be overcome: the beginner can be persuaded that it makes sense, after all, to speak of the number of natural numbers. Once his initial prejudice has been overcome, the next stage is to convince the beginner that there are distinct [infinite] cardinal numbers: not all infinite totalities have as many members as each other. When he has become accustomed to this idea, he is extremely likely to ask, 'How many transfinite cardinals are there?'. How should he be answered? He is very likely to be answered by being told, 'You must not ask that question'. But why should he not? If it was, after all, all right to ask, 'How many numbers are there?', in the sense in which 'number' meant 'finite cardinal', how can it be wrong to ask the same question when 'number' means 'finite or transfinite cardinal'? A mere prohibition leaves the matter a mystery. It gives no help to say that there are some totalities so large that no number can be assigned to them. We can gain some grasp on the idea of a totality too big to be counted . . . but once we have accepted that totalities too big to be counted may yet have numbers, the idea of one too big even to have a number conveys nothing at all. And merely to say, 'If you persist in talking about the number of all cardinal numbers, you will run into contradiction', is to wield the big stick, but not to offer an explanation.[26]

However, I think the parallel is questionable. It is true that we only get the indefinitely extensible series of transfinite cardinals up and running in the first place by first insisting on one-one correspondence between concepts as necessary and sufficient for sameness, and hence existence, of cardinal numbers in general—not just in the finite case— and that the conception of *cardinal number* as embracing both the finite and the spectacular array of transfinite cases thus only arises in the first place when it is taken without question that concepts in general—or at least *sortal*

[26] Dummett (1991), pp. 315-316

concepts in general: concepts that can sustain determinate relations of one-one correspondence—have cardinal numbers, identified and distinguished in the light of those relations. That is how the intuitive barrier to the question, how many natural numbers are there, is overcome. And it is also true that that at least loosens the lid on Pandora's box: for the intuitive barrier to the question, how many *cardinal* numbers are there is thereby overcome too. But loosening the lid isn't enough to trigger paradox. Hume's principle, identifying the cardinal numbers associated with sortal concepts in general just when those concepts are bijectable, encapsulates exactly the "resistance-overcoming" move that Dummett is taking about. And it generates, indeed, not merely a cardinal number of cardinal numbers but the universal number "Anti-zero", the number of absolutely everything that there is. But it does *not* spawn any paradox, as far as it goes. It is a consistent principle; at least, it is consistent in classical second-order logic. To get the paradox — Cantor's paradox — out of the notion of cardinal number that Hume's principle characterises, we need to embed it in a set-theory containing the associated principles sufficient to generate Cantor's theorem itself: unrestricted Union, an exemplar set for any given set of cardinals, and a set of all cardinals. None of that baggage is entailed just by the assumption that every sortal concept has a cardinal number, identified and distinguished from others by relations of one-one correspondence.

Moreover, the notion of cardinal number is not needed at all to spring that paradox. Given only a universal set, and unrestricted power set, standard moves in naive set-theory will allow us to prove both that its power set is injectable into the universal set (since the former is a proper subset of the latter) and that the there can be no such injection (via the diagonalisation in Cantor's theorem.) This is already a paradox. But it is a paradox for the (naïve) notion of *set*. *Cardinal number*, as extended into the transfinite via a criterion of one-one correspondence, is not in play. Someone could reject that extension and still have to confront the antinomy. The core of Cantor's paradox can indeed be assumed under our template for a paradox of indefinite extensibility: simply take P as *object* (or *self-identical*), ΠQ as *the Qs compose a set* and F as the power-set operation. Consider any such *Π* concept, *Q*. The reasoning of Cantor's theorem shows that some of the members of *FQ* cannot be instances of *Q*. This immediately gives a contradiction when *P* itself is taken to be Π, i.e. when we assume a universal set. But no assumptions about *cardinal number* are involved. It is true that, as illustrated earlier, *cardinal number* is indefinitely extensible with respect to *set* when the appropriate assumptions about sets —union, power and replacement — are made, and that this is enough for a paradox of indefinite extensibility if *cardinal number* is itself assumed to determine a set. But this should impress as a frame-up, rather than an insight. The real problem is with the set-theoretic assumptions involved.

Notice, incidentally, that if we deny that *set*, and *cardinal number* themselves determine sets, then we obtain — or at least I know of no reason to doubt that we obtain — examples of the possibility shortly canvassed earlier: concepts that are indefinitely extensible but with whose indefinite extensibility no paradox is (so far as one can see) associated. The philosophical justifiability of that denial is, naturally, entirely another matter.

7. Basic Law V

If the foregoing is correct, the cases of two of the 'principal suspects', *ordinal number* and *cardinal number,* are different. The former is unquestionably guilty as charged: *ordinal number*, as intuitively understood, is essentially reflexively indefinitely extensible and thereby paradox generating; but the jury should find the charge against *cardinal number* unproven. When comprehension principles are accepted for the ordinals that both ensure that every well-ordered collection has an ordinal and provide for unlimited applicability of successor and limit, *ordinal number* is essentially susceptible to a paradox of indefinite extensibility *qua* satisfying the relevant trigger concept, Π. But when comprehension for the cardinals is determined by Hume's Principle, it takes set-theoretic assumptions to make a case that *cardinal number* is indefinitely extensible, and further set-theoretic assumptions to make a paradox out of that. These assumptions have no evident intrinsic connection with *cardinal number*.

So what, finally, about *value-range* as it features in Law V? Is it appropriate — insightful— in the light of the account of indefinite extensibility now on the table, and the connection of its reflexive variant with paradox, to attribute the antinomy that Russell discovered to the indefinite extensibility of the notion that Law V characterises?

Well, there are some subtleties here, and a major unresolved issue. Let's start by noting that there is certainly *a* paradox of indefinite extensibility in the offing. Here is how it goes. Restrict attention to the case of value -ranges whose domains are concepts and values truth-values—i.e. to the case of *extensions of concepts*— so that we have in effect this special case of the axiom:

$$(\forall P)(\forall Q)(\{x{:}Px\}=\{x{:}Qx\} \leftrightarrow (\forall x)(P\mathrm{x} \leftrightarrow Q\mathrm{x}))$$ [27]

Extensionality and Naïve Comprehension can be read off straight away: extensions are identical just when their associated concepts are co-extensive; and every concept has one. (Proof: take '*P*' for '*Q*', detach the left-hand-side of the biconditional, and existentially generalise on one occurrence of '$\{x{:}Px\}$'.) So *absolutely any* concept of extensions is associated with its own extension. Take *P* then as *extension* itself, and Π as *has an extension*. Let *Q* be any subconcept of *P*. By Law V, *Q* has an extension. Define membership in one of the natural ways.[28] Consider the concept: *Qx and not xεx*. Call this concept *Q**. Form its extension, *q**. Choose this for *FQ*. Suppose *Qq**. Do we have *q*εq**? If so then, *q** falls under *Q** and is thereby a *Q* that is not a member of itself. But, by the definition of q*, *Qq* and not q*εq** is in turn sufficient for q* to be a member of itself. Contradiction. So not *Qq** . Take *Q'* as the concept: *Qx V x=q** . . .

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[27] — frequently, though strictly incorrectly, represented as Law V in contemporary discussion.

[28] For instance, stipulate that *x* is a member of *y* just if *x* satisfies every *P* of which *y* is the extension; or that *x* is a member of *y* just if *x* satisfies some *P* of which *y* is the extension. (Note that the former, though not the latter, will have the effect that *x* will be a member of *y* if *y* is not an extension; but they will coincide if we restrict our ontology to the items characterised by Law V as—after the stipulation he introduces in *Grundgesetze* §10 to address the analogue of the Caesar problem for the True and the False—Frege in effect does.)

Referring back to the three conditions listed in section 3 for our initial, relativised notion of indefinite extensibility, the foregoing completes a case for saying that *extension* is indefinitely extensible with respect to *has an extension*. Paradox is then immediate when we reflect that by a special case of Law V, we should intuitively have that ΠP, i.e. that *extension* itself has an extension, and so is *reflexively* indefinitely extensible with respect to *has an extension*. (Compare: that there is a set of all sets.)

Since Law V provides us with a singular-term forming operator on concept-expressions whose sense is effectively that of "The extension of . . ", we can run the foregoing paradox in *Grundgesetze* if we take the concept of an extension to be captured by: *for some F, x = the extension of F*. But although it smells pretty similar, this is not quite the paradox that Russell discovered. Paradoxes of indefinite extensibility, as now understood, turn essentially on *reflexive* indefinite extensibility: on the application of the principle of extension, *F*, to the indefinitely extensible concept *P* itself— an application made possible by *P*'s satisfaction of the higher-order trigger concept, Π. The paradox just adumbrated has exactly that shape, but the indefinite extensibility of *extension* (*value-range*) doesn't feature in the reasoning from Law V that Russell found — or at least, that Frege took him to have found. The key resource for that reasoning is simply the license, granted by Law V, to take it that every monadic open sentence expressible in *Grundgesteze* that has an objectual argument place has an extension, and hence in particular that *x is not self-membered* has an extension. The assumption that *extension* has an extension is not at work in the Russellian brew.

A reminder may be helpful of how the brewing goes. Derive naïve abstraction for extensions $(\forall P)(\exists y)(y = \{\mathrm{x}:Px\})$

from Law V in the manner adumbrated above and then instantiate *P* to (this version, e.g, of) the concept of non-self-membership:

$$(\exists Q)(z = \{\mathrm{x}:Qx\} \ \& \ {\sim}Qz),$$

Apply naïve abstraction to that to obtain Russell's rogue extension:

$$\{z: (\exists Q)(z = \{\mathrm{x}:Qx\} \;\&\; {\sim}Qz)\}$$

and call this object **r**. Suppose now that ***r*** satisfies the condition on its own members, i.e., is a member of itself:

$$(\exists Q)(\boldsymbol{r} = \{x:Qx\} \;\&\; {\sim}Q\boldsymbol{r})$$

Let P be a witness of this existential. Since, by Law V, any concept of which **r** is the extension is co-extensive with P, it follows that non-self-membership as defined above is co-extensive with P,[29] and hence, since ${\sim}P\boldsymbol{r}$, that

$${\sim}(\exists Q)(r = \{\mathrm{x}:Qx\} \;\&\; {\sim}Qr),$$

i.e. that **r** fails to satisfy the condition on its own members. (That's the step at which, egregiously impredicatively, we assume the Russellian condition to lie within the range of its own existential quantifier.) It follows (classically) that ***r*** falls under every concept Q of which it is the extension, and hence that it satisfies the condition on its own members after all . . .

Again: this reasoning does not fit the template for a paradox of indefinite extensibility for *extension*. In the presence of Law V in full generality, *extension* is indeed, intuitively, reflexively indefinitely extensible, but the paradox to which Russell drew Frege's attention is not the paradox associated with that point.

Still, as some readers may be impatient to observe,[30] it is possible to present Russell's paradox as a paradox of indefinite extensibility by exactly the standards of our template for such paradoxes. Only the concept whose reflexive indefinite extensibility it exploits is not that of *extension* as such but rather: *falls under no concept of which it is the extension*. Take this concept

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[29] As the reader will appreciate, it is at this point that the attempt to derive a corresponding contradiction from Hume's Principle is thwarted.

[30] Thanks to Toby Meadows here.

for P and take Π as *has an extension*. And let $F(Q)$ simply be the extension of the concept, Q. Now Law V gives that ΠP. And Russell's paradox, now explicitly wearing the face of a paradox of indefinite extensibility, ensues.

Here is the detail. We need to show that, in the presence of Law V, these selections for P, Π, and F deliver each of the conditions proposed in section 3 above for P to be indefinitely extensible with respect to Π, viz. that for our chosen function F

(1) FQ falls under the concept P,
(2) It is not that case that FQ falls under the concept Q, and
(3) $\Pi Q'$, where Q' is the concept instantiated just by FQ and by every item which instantiates Q

(3) is immediate from Law V. For (2), suppose for reductio that $Q(FQ)$. Then, since Q is any sub-concept of P, we have $P(FQ)$. So by our choice for P, FQ falls under no concept of which it is the extension. So (2) FQ doesn't fall under Q. So that is one concept of which FQ is the extension but under which it does not fall. It follows by Law V that it fails to fall under any concept of which it is the extension, and hence (1) that $P(FQ)$.

Alright. So should we now accept that the Dummettian diagnosis of the serpent's ingress with which we started is correct?

Well, it is doubtless true that Frege had "no glimmering of a suspicion" of the notion of (reflexive) indefinite extensibility, and has consequently overlooked that his Law V, in conjunction with the proof theory and definitional (open-sentence forming) resources of the underlying logic of *Grundgesetze*, allows us to introduce (the extensions of) a whole range of reflexively indefinitely extensible concepts and provides deductive resources sufficient for the derivation of the associated paradoxes. However, I think Dummett's account should impress as, so far, a much more tendentious explanation of the roots of the paradoxes in *Grundgesetze* than the corresponding diagnosis in the case of *ordinal* and Burali-Forti.

The Burali-Forti paradox flows directly from comprehension principles that go right to the heart of the intended notion of *ordinal number*. With *cardinal number*, by contrast, or so I argued, there is no such direct connection: a paradox of indefinite extensibility can indeed be manufactured for the notion, but the apparatus required to do that involves significant set-theoretic postulates (including in particular the assumption that the cardinals compose a set) that have no intrinsic connection with the idea of cardinal number *per se*. And the case of the paradoxes associated with Law V is arguably closer to the latter situation. To stress: the paradoxes in *Grundgesetze* arise from a *co-operation* between the principle of objectual comprehension encoded in Law V and the *impredicative* principles of conceptual, or functional, comprehension that are inexplicit in Frege's own presentation but crucial to the intended functioning of his system. Law V encodes the most straightforward possible view—absolutely integral to Frege's philosophy of mathematics and his treatment of mathematical existence —of the relation between concepts and their associated logical objects. But the propensity of this straightforward view to issue in indefinitely extensible populations of mathematical objects entirely depends upon the collateral repertoire of concepts that Frege seemingly unhesitatingly plunged into—a repertoire that incorporated unconstrained use of formulas involving quantification over all concepts, and allowed any such formulas with free objectual argument places in turn to determine concepts falling within their own range of quantification. When, by contrast, Law V is taken in conjunction with predicative systems of higher-order logic, no indefinitely extensible concepts of the objects it concerns can be formulated and the resulting systems are consistent.[31]

So we have a competitor to Dummett's diagnosis, viz. that Frege fell into paradox because he failed to think through the implications, in the presence of Law V, of the full

[31] A result due to Heck (1996). For a valuable overview of potency and consistency issues for predicative second-order theories based on Law V, see Burgess (2005) ch. 2.

repertoire of open sentences on which the higher-order quantifiers in *Grundgesetze* are permitted to generalise— failed, if you like, to reckon with the expressive resources, and especially those of diagonalisation, that come with classical, impredicative higher-order logic. He simply "didn't think of that kind of case".

Which is the better account? Well, someone who sides with Frege in taking it that classical impredicative higher-order logic is nonetheless *the correct* higher-order logic will have no option but to assimilate Russell's paradox to Burali-Forti's: each will be correctly viewed as a paradox of indefinite extensibility, properly so described, flowing directly from comprehension principles that are integral to the species of objects concerned. There will then be no alternative but to conclude that the simple correlation between concepts (more generally, functions) and objects postulated by Law V encapsulates a conception of mathematical ontology that was not merely absolutely integral to Frege's own logicism— a conception whereby the mathematical objects of arithmetic and analysis are simply the logical objects that are the Fregean surrogates of functions—but also incoherent at its core. And this of course was Frege's own reaction. So conceived, the paradox does indeed go right to the heart of his vision of the subject matter of mathematics. That is why his reaction to it was eventually one of despair.

But there is the alternative: to question whether the kind of generality that is the legitimate focus of higher-order logic is correctly implemented by the unrestrictedly impredicative system that Frege invented. It is familiar that reservations about this tend to be inaudible to one who thinks of the range of the higher-order quantifiers as a fixed comprehensive universe, either of sets or of set-like entities, and that they tend to seem urgent to one who thinks rather of such quantification as essentially answerable to the satisfaction-conditions of formulable, intelligible open-sentences. The division corresponds roughly to that between those who sympathise with Quine's famous jibe about higher-order logic, that it is essentially set-theory in "sheep's clothing" and have no time for the predicativist restrictions on set-theory itself that would pre-empt its mathematically more exotic reaches, and those who recognise that higher-order logic's claim to be *logic* rests squarely on its capacity to constrain the scope of its quantifiers to generalisation over predications, atomic and complex, that can in principle feature in the thought and inferential practices of a rational agent.[32]

I know of no basis for attributing the latter type of view to Frege. But there is a case for thinking that any serious logicism *must* work with a conception of higher-order logic of this broad stripe if a successful execution of the technical part of its programme is to carry the epistemological significance that is traditionally intended. It is therefore an awkward fact, from the point of view of a sympathiser with Frege's project, that, syntactically viewed, the lowest order of impredicative comprehension sufficient for the paradoxes is also needed for the derivation from Law V of Peano arithmetic and real analysis.[33] Had Frege anticipated and endorsed predicativism in his philosophy of logic, the formal project of *Grundgesetze* would thus have been curtailed in any case. And while there are forms of impredicative comprehension that can be consistently adjoined to Law V in company with the standard proof theory of classical higher-order logic, these too stop short of generating the repertoire of concepts needed for the recovery of Frege's Theorem and the axioms for a complete ordered field.[34]

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[32] The set-theoretic interpretation of higher-order logic is of course entrenched. The foregoing kind of reservation about it and alternative approach is represented in work of Hale —see his (2013) and forthcoming, and ch. 8 of Hale (2103a), —and the present author (2007), drawing on Rayo and Yablo (2001)

[33] Russell's paradox as sketched above relies on ∑1-1 comprehension, but may equally well be accomplished using the variant characterisation of membership given at n. 28 above and ∏1-1 comprehension. ∏1-1 comprehension is needed for the deduction of the Peano Postulates from Hume's principle in second-order logic (specifically, for the proof of the Induction Axiom — see Heck (2011) at p. 289ff—and, if Frege's own definitions of the arithmetical primitives are used, for the proof that every number has a successor; see Linnebo (2004) —but no additional comprehension is needed. I believe, but have not at the time of writing confirmed, that nothing above ∏1-1 comprehension is needed for existing abstractionist recoveries of Real Analysis either in the style of Hale [2000] or the more Dedekindian approach of Shapiro [2000].

[34] Wehmeier [1999] shows that Law V plus classical higher-order logic with Δ1-1 comprehension gives a consistent system, (where *Qx* is Δ1-1 comprehensible just in case equivalent both to some ∑1-1 comprehensible predicate *and* to some ∏1-1 comprehensible predicate.) The system he considers treats the extension-forming operator as a functor attaching only to variables. A consistency result for a somewhat stronger system, in which the extension operator is applied also to formulas, is obtained in Ferreira and Wehmeier (2002). Paradox is avoided in these systems because self-membership is not Δ1-1 comprehensible—recall that, as there remarked, the ∑1-1 and ∏1-1 formulations of membership bruited in

As remarked earlier, Dummett's writings on this topic are shot through with the idea that the contradictions are the symptom of a deeper philosophical mistake, that Russell's paradox is, as it were, a carbuncle on the face of an edifice that betrays a deeper underlying malaise. For Dummett, the indefinite extensibility of fundamental mathematical domains is a philosophically vital fact about them, and one gets the impression almost that he regarded the paradox as a fitting nemesis for Frege's failure to understand and acknowledge this fact. (Though he nowhere says what Frege should have done differently if he had recognised the fact, nor how it would have helped.) The upshot of our discussion emerges as that this diagnosis is premature. First, it is, in any case, *reflexive* indefinite extensibility that is paradox-spawning, not indefinite extensibility as such. But second—the principal point—the objects that Law V introduces compose the instances of a reflexively indefinitely extensible concept only when the underlying logic avails itself of impredicative forms of comprehension whose consistency with its status as logic proper *is an unsettled philosophical issue*. In our present state of understanding, then, we should not say that Frege's most fundamental error was to overlook the indefinite extensibility of *value-range* as characterised by Law V, but merely that he failed to recognise that his conception of the nature of the objects of arithmetic and analysis, encoded in Law V, would not cohere with the unrestricted use of the higher-order definitional and proof theoretic resources that he needed to obtain the fundamental laws of arithmetic and analysis from it.

8. Coda

It is wholly understandable that, in his historical context, Frege failed to perceive, let alone address the questions concerning the epistemological status of his logic, and its legitimate expressive resources, which the contribution made by impredicative comprehension in the generation of paradox brings to the centre of the stage. Modern neo-Fregeans have perhaps less

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n. 28 above are not equivalent. It turns out, however, that these systems, even if their comprehension principles could be philosophically motivated, are too weak even for the recovery of primitive recursive arithmetic. For details see Cruz-Felipe and Ferreira (2015).

excuse. It is, of course, of considerable interest, both technically and philosophically, that weakening the powers of objectual comprehension encoded in Law V by replacing it with selected, theory-specific abstraction principles while retaining impredicative higher order proof-theoretic resources can provide deductively adequate foundations for arithmetic and analysis. But the philosophical significance of these results continues to depend upon the epistemological standing of the underling impredicative higher-order logic; and of all the philosophical issues arising in the intensive debates about neo-Fregeanism over the last 30 years, this one surely scores maximum points for the simultaneous combination of urgency and neglect.[35] The question, for any properly logicist interpretation of higher-order logic, is which (if any) kinds of impredicative comprehension keep us within the bounds set by the vague notion of intelligible predication and to what (if any) extent can the exploitation of the impredicative resources technically required for logicist foundations respect those bounds?

It has become customary to look at the issues here through the lense of the purely syntactic classifications delivered by the Π- and Σ- analytical hierarchies, with escalation of the indices somehow taken as representing decreasing (epistemic? metaphysical?) modesty or increasing risk. Maybe there is something to that instinct. But the natural starting point for any investigation, it seems to me, has to be not syntax but meaning. We need to consider how quantified sentences get content in the first place, and here there is a very natural if inchoate thought: that the truth-conditions of quantified sentences (of any order) must somehow be grounded in the distribution of truth-values across the entire range of their admissible instances, and hence that open-sentences formed from quantified sentences by leaving free variables in places where expressions for their instances may stand, can have determinate satisfaction-conditions only to the extent that this basing constraint is respected.

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[35] One important exception to the trend is Øystein Linnebo (2009). See also Linnebo (2010).

This thought, that the truth-conditions of intelligible quantified statements of any order require to be grounded in those of statements of the immediately preceding order, obviously needs refinement. What, for instance, is the relevant relation of grounding, or basing? And what about cases where the range of the quantifier outruns any conceivable expressive resources? Still, it contains, I believe, the kernel of the most basic misgiving about higher-order impredicativity. The question is therefore whether, properly developed and understood, the need for grounding does not actually require predicativism but can be respected by certain kinds of impredicative case.[36] Syntactically viewed, as noted, the levels of impredicative comprehension needed for the development of arithmetic and analysis are no more modest than those required for the derivation of Russell's paradox. Might there nevertheless be a philosophically significant line to draw between them that we have so far missed? It is not merely the assessment of the import of neo-Fregeanism's technical achievements that awaits an answer. Until we have one, there is no knowing for sure that there is not some well-motivated, albeit non-syntactic, constraint on impredicative comprehension that stabilises the project of *Grundgesetze*.[37]

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[36] Something very similar to this is one upshot of the proposals developed by Øystein Linnebo in the papers cited in n. 35. Linnebo outlines a general theory of grounding by stages, embracing both concepts and abstract objects, that aims to underwrite a restriction on classical comprehension axioms sufficient to pre-empt paradox (and indeed solve the Bad Company problem for abstractionism more generally) but which is potentially more generous than a requirement of simple (or ramified) predicativity. His proposals, however are motivated by metaphysical considerations rather than the kind of semantic considerations gestured at in the text, and are neutral between objectual and 'conceptual' (anti-Quinean) understandings of higher-order generality. I hope to treat of them in further work.

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