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STAR COMPLEMENTS IN REGULAR GRAPHS: OLD AND NEW RESULTS

Peter Rowlinson[1]
Mathematics and Statistics Group
Department of Computing Science and Mathematics
University of Stirling
Scotland FK9 4LA

Behruz Tayfeh-Rezaie[2]
School of Mathematics
Institute for Research in Fundamental Sciences
PO Box 19395-5746
Tehran
Iran

Abstract

We survey results concerning star complements in finite regular graphs, and note the connection with designs and strongly regular graphs in certain cases. We include improved proofs along with new results on stars and windmills as star complements.

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[1]Corresponding author, tel.:+44 1786 467468; fax +44 1786 464551; email: p.rowlinson@stirling.ac.uk
[2]Email: tayfeh-r@mail.ipm.ir

1 Introduction

Let G be a finite simple graph with vertex-set $V(G) = \{1, 2, \ldots, n\}$ and (0,1)-adjacency matrix $A = (a_{ij})$. The eigenvalues of A are independent of the vertex-ordering and are therefore called *eigenvalues of* G. For such an eigenvalue μ, let $\mathcal{E}(\mu)$ denote the eigenspace $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mu\mathbf{x}\}$. Let P be the matrix of the orthogonal projection of $\mathbb{R}^n$ onto $\mathcal{E}(\mu)$ with respect to the standard orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ of $\mathbb{R}^n$. Then $\mathcal{E}(\mu)$ is the column space of P, and so there exists $X \subseteq V(G)$ such that the vectors $P\mathbf{e}_j$ $(j \in X)$ form a basis for $\mathcal{E}(\mu)$. Such a set is called a *star set* for μ in G. (The terminology reflects the fact that the vectors $P\mathbf{e}_j$ $(j = 1, \ldots, n)$ form a *eutactic star*, as defined by Seidel [38].)

We write $\overline{X}$ for the complement of X in $V(G)$, and we write $G - X$ for the subgraph of G induced by $\overline{X}$. If X is a star set for the eigenvalue μ then $G - X$ is said to be a *star complement* for μ in G. (Such graphs are called *μ-basic* subgraphs in [15].) It is clear from the definitions that star sets and star complements exist for any eigenvalue of any graph. A database of about 1500 examples is described in [9], and a survey of star complements appears in [30]. It is observed in [15] (and attributed to S. Penrice) that if μ is an eigenvalue of a connected graph G then G has a connected star complement H for μ (see also [32, Theorem 2.4]). Moreover, H may be taken to contain any connected induced subgraph of G that does not have μ as an eigenvalue [12, Proposition 1.1].

In practice, it is often convenient to use the characterization of star complements given by condition (iii) in the following result.

Theorem 1.1 [10, Theorem 7.2.9] *Let G be a graph, let $X \subseteq V(G)$ and let μ be an eigenvalue of G with multiplicity k. Then the following statements are equivalent:*

(i) $\{P\mathbf{e}_j : j \in X\}$ *is a basis of* $\mathcal{E}(\mu)$,

(ii) $\mathbb{R}^n = \mathcal{E}(\mu) \oplus \mathcal{V}$, *where* $\mathcal{V} = \mathrm{span}\{\mathbf{e}_j : j \in \overline{X}\}$,

(iii) $|X| = k$, *and μ is not an eigenvalue of* $G - X$.

The Interlacing Theorem [27, Theorem 34.2.2] ensures that, when a vertex is deleted from a graph, the eigenvalue multiplicities change by 1 at most, and so (in the situation of Theorem 1.1) deletion of any r vertices from X $(0 < r < k)$ results in a graph with μ as an eigenvalue of multiplicity $k - r$.

Before discussing an example, we introduce some more notation. In the literature, we find complementary definitions of the Clebsch graph, and to avoid this difficulty we write Cl_5 for the unique strongly regular graph with parameters $(16, 5, 0, 2)$ and Cl_{10} for its complement. The same is true of the Schläfli graph, and so we write Sch_{10} for the unique strongly regular graph with parameters $(27, 10, 1, 5)$, and Sch_{16} for its complement. Similarly, we write McL_{112} for the McLaughlin graph [26], and McL_{162} for its complement. The graph McL_{112} is the unique strongly regular graph with parameters $(275, 112, 30, 56)$ (see [5]).

We write '$u \sim v$' to mean that vertices u and v are adjacent, and we let $(a_{ij}^{(s)}) = A^s$ $(s \in I\!N)$. The join $G_1 \nabla G_2$ is the graph obtained from the disjoint graphs G_1, G_2 by joining every vertex of G_1 to every vertex of G_2.

Example 1 The graph Cl_5 may be constructed as follows [5, p. 35]. The vertices are the even subsets of a 5-set, and two even sets are adjacent if and only if their symmetric difference has size 4. Thus the vertex set has a partition $X_0 \dot{\cup} X_2 \dot{\cup} X_4$, where the sets in X_i have size i $(i = 0, 2, 4)$, X_2 induces a Petersen graph, and $X_0 \dot{\cup} X_4$ induces a star $K_{1,5}$. The spectrum of Cl_5 is $5, 1^{10}, (-3)^5$, and 1 is not an eigenvalue of $K_{1,5}$; hence X_2 is a star set for the eigenvalue 1. Similarly, X_4 is a star set for -3 and X_0 is a star set for 5. $\square$

In Example 1, the vertex set is partitioned by the star sets X_0, X_2, X_4, and so these sets are said to form a *star partition.* In fact, every graph has a star partition [10, Theorem 7.1.3]. For a discussion of star partitions in the context of the graph isomorphism problem, see [10, Chapter 8].

The following result, known as the The Reconstruction Theorem (and its converse), is fundamental to the theory of star complements. This application of the Schur complement [27, p.17] in graph theory was noted independently by Ellingham [15] and Rowlinson [28] in 1993.

Theorem 1.2 [13, Theorem 5.1.7] *Let X be a set of k vertices in the graph G and suppose that G has adjacency matrix $\begin{pmatrix} A_X & B^\top \\ B & C \end{pmatrix}$, where A_X is the adjacency matrix of the subgraph induced by X. Then X is a star set for μ in G if and only if μ is not an eigenvalue of C and*

$$\mu I - A_X = B^\top (\mu I - C)^{-1} B. \qquad (1)$$

In this situation, $\mathcal{E}(\mu)$ consists of the vectors $\begin{pmatrix} \mathbf{x} \\ (\mu I - C)^{-1} B\mathbf{x} \end{pmatrix}$ $(\mathbf{x} \in I\!R^k)$.

The columns $\mathbf{b}_u$ $(u \in X)$ of the matrix B are the characteristic vectors of the H-neighbourhoods $\Delta_H(u) = \{v \in V(H) : u \sim v\}$ $(u \in X)$. Thus Theorem 1.2 shows that any graph is determined uniquely by an eigenvalue μ, a star complement $H = G - X$ and the H-neighbourhoods of vertices in X. This establishes the role of a single eigenvalue in determining the structure of a graph.

We write $t = |\overline{X}|$ $(= n - k)$ and define a bilinear form on $I\!R^t$ by:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\top (\mu I - C)^{-1} \mathbf{y} \quad (\mathbf{x}, \mathbf{y} \in I\!R^t).$$

By equating entries in (1) we see that X is a star set for μ if and only if μ is not an eigenvalue of $G - X$ and the following conditions hold:

$$\langle \mathbf{b}_u, \mathbf{b}_u \rangle = \mu, \quad \text{for all} \;\; u \in X, \qquad (2)$$

and for distinct $u, v \in X$,

$$\langle \mathbf{b}_u, \mathbf{b}_v \rangle = -1 \text{ if } u \sim v, \quad \langle \mathbf{b}_u, \mathbf{b}_v \rangle = 0 \text{ if } u \not\sim v. \tag{3}$$

We call (3) the *compatibility condition.* In view of Equations (2) and (3), we have:

Proposition 1.3 [10, Theorem 7.6.2] *Let X be a star set for μ in G and let $H = G - X$.*
(i) *If $\mu \neq 0$ then the H-neighbourhoods of vertices in X are non-empty.*
(ii) *If $\mu \neq -1, 0$ then the H-neighbourhoods of vertices in X are distinct and non-empty.*

In other words, if $\mu \neq 0$ then $\overline{X}$ is a dominating set, and if $\mu \neq -1, 0$ then $\overline{X}$ is a location-dominating set in G, as defined in [40]. We say that $\overline{X}$, with vertices $1, 2, \ldots, t$, is a k-location-dominating set if

$$(a_{u1}^{(k)}, a_{u2}^{(k)}, \ldots, a_{ut}^{(k)}) \;\neq\; (0, 0, \ldots, 0) \;\text{ for all } u \in X,$$

and for any pair u, v of vertices in X,

$$(a_{u1}^{(k)}, a_{u2}^{(k)}, \ldots, a_{ut}^{(k)}) \;\neq\; (a_{v1}^{(k)}, a_{v2}^{(k)}, \ldots, a_{vt}^{(k)}).$$

Properties of k-location-dominating star complements in regular graphs are investigated in [25]. An earlier result concerning the dominating property of star complements in regular graphs is the following (see [10, Corollary 7.6.8]).

Theorem 1.4 [29, Theorem 3.4(ii)] *Let G be a connected regular graph in which X is a star set for the eigenvalue $\mu \neq -1, 0$. If the dominating set $\overline{X}$ is a minimal dominating set, and if the star complement $G - X$ has no isolated vertices, then G is the Petersen graph.*

It follows from Proposition 1.3 that if $\mu \neq -1, 0$ then $n < t + 2^t$. Thus there are only finitely many graphs with an eigenspace $\mathcal{E}(\mu)$ ($\mu \neq -1, 0$) of prescribed codimension, and this observation is the basis for characterizing graphs by star complements, as documented in [30]. For example, McL_{112} is the largest connected graph with $K_{1,16} \,\dot{\cup}\, 6K_1$ as a star complement for the eigenvalue 2 [36, Theorem 3.1]. As an example of a more general characterization we have, for odd $t > 3$: G has the cycle C_t as a star complement for -2 if and only if G is the line graph of a Hamiltonian graph of order t [1, Theorem 2.4]. A sharp upper bound for n as a quadratic function of t (for $\mu \neq -1, 0$) is given in Section 5. Consideration of K_n and $K_2 \,\dot{\cup}\, (n-2)K_1$ shows that n cannot be bounded in terms of t when $\mu = -1, 0$.

To describe all the graphs with H as a star complement, we solve Equation (1) for A_X, B and μ, given C (the 'general problem'). Since μ is necessarily an eigenvalue of a one-vertex extension of H, there are only finitely many possibilities for μ. The 'restricted problem' is to find the solutions A_X, B of (1) for a given matrix C and a given eigenvalue μ, a process generally called the *star complement technique* (see [13, Chapter 5]). To describe all the graphs with H as a star complement for $\mu \notin \{-1, 0\}$, it suffices to determine those graphs for which X is maximal, since any graph with H as a star complement for μ is an induced subgraph of such a graph. To construct these graphs, we introduce the *compatibility graph* (or *extendability graph*) $\Gamma(H, \mu)$ defined as follows: the vertices of $\Gamma(H, \mu)$ are the $(0, 1)$-vectors $\mathbf{b}$ in $I\!R^t$ such that $\langle \mathbf{b}, \mathbf{b} \rangle = \mu$, and distinct $\mathbf{b}$, $\mathbf{b}'$ are joined by an edge if $\langle \mathbf{b}, \mathbf{b}' \rangle \in \{-1, 0\}$. A graph G with a maximal star set X for μ such that $G - X = H$ now corresponds to a maximal clique in $\Gamma(H, \mu)$. Accordingly the compatibility graph is well suited to a computer implementation of the star complement technique (cf. [15, Algorithm 2.4]). The graph $\Gamma(C_5, 1)$ is illustrated in [13, Section 5.1].

In addition to the general problem and the restricted problem, one can consider the problem of finding all solutions A_X, C, μ of (1) for a given matrix B. This problem is solved in [8] in the case that B is an identity matrix. In this situation, if G is connected then one of the following holds: (a) $\mu = \pm 1$ and $G = K_2$, (b) $\mu = 0$ and $G = C_4$, (c) $\mu = 1$ and G is the Petersen graph (cf. Theorem 1.4). Here we have a special case of a *uniform* star set as defined in [33]: the star set X is said to be uniform if all vertices in $\overline{X}$ have the same number of neighbours in X. Thus if G is regular and X is uniform then the star complement $G - X$ is also regular; cubic graphs satisfying these conditions are classified in [33, Section 3] (see also [10, Chapter 6]) .

If G is r-regular and $\mu \neq r$ then the all-1 vector $\mathbf{j}_n$ is orthogonal to μ; in other words, μ is a non-main eigenvalue. From the description of $\mathcal{E}(\mu)$ in Theorem 1.2, we have the following result, where we write $\mathbf{j}$ for $\mathbf{j}_t$.

Proposition 1.5 [11, Proposition 0.3] *The eigenvalue μ is a non-main eigenvalue if and only if*

$$\langle \mathbf{b}_u, \mathbf{j} \rangle = -1 \quad \text{for all } u \in X. \tag{4}$$

To find the regular graphs with H as a star complement for μ, it clearly suffices to consider the subgraph $\Gamma^*(H, \mu)$ of $\Gamma(H, \mu)$ induced by those vectors $\mathbf{b}$ for which $\langle \mathbf{b}, \mathbf{j} \rangle = -1$; this is called the *non-main compatibility graph* in [14]. For example, $\Gamma^*(C_5, 1) = K_5$, and the unique regular graph with C_5 as a star complement for 1 is the Petersen graph.

2 Stars as star complements

We shall require the following observation:

Lemma 2.1 *Let G be a connected r-regular graph with μ $(\neq r)$ as an eigenvalue of multiplicity k. Suppose that $|V(G)| = r + k + 1$. Then*

$$r - \mu^2 - \tfrac{k}{r}\mu^2 - 2\mu \geq 0,$$

with equality if and only if G is strongly regular.

Proof. Note that neither G nor $\overline{G}$ is complete. Let $\theta_1, \ldots, \theta_r$ be the eigenvalues of G other than μ and r. We have

$$\sum_{i=1}^{r} \theta_i + k\mu + r = 0 \quad \text{and} \quad \sum_{i=1}^{r} \theta_i^2 + k\mu^2 + r^2 = (1+k+r)r.$$

It follows that if $\overline{\theta} = \frac{1}{r}\sum_{i=1}^{r} \theta_i$ then

$$\sum_{i=1}^{r}(\theta_i - \overline{\theta})^2 = \sum_{i=1}^{r} \theta_i^2 - r\overline{\theta}^2 = k(r - \mu^2 - \tfrac{k}{r}\mu^2 - 2\mu),$$

and this establishes the inequality. Equality holds if and only if $\theta_i = \overline{\theta}$ $(i = 1, \ldots, r)$, equivalently G has just three distinct eigenvalues. By [13, Theorem 1.2.20], a non-complete connected regular graph is strongly regular if and only if it has exactly three distinct eigenvalues. □

The strongly regular graphs that arise in Lemma 2.1 are either conference graphs or graphs of Latin square type (see [5, Proposition 8.14]). Here we first use Lemma 2.1 to strengthen Theorem 2.2 of [34]:

Theorem 2.2 *If the r-regular graph G has $K_{1,s}$ $(s > 1)$ as a star complement for μ then one of the following holds:*

(a) $\mu = \pm 2$, $r = s = 2$ *and G is a 4-cycle;*
(b) $\mu = \frac{1}{2}(-1 \pm \sqrt{5})$, $r = s = 2$ *and G is a 5-cycle;*
(c) $\mu \in I\!N$, $r = s$ *and G is strongly regular with parameters* $((\mu^2+3\mu)^2, \mu(\mu^2+3\mu+1), 0, \mu(\mu+1))$.

Proof. By Theorem 1.1(iii) we have $\mu \neq 0$, and so G is connected by Proposition 1.3(i). Let $|V(G)| = n$. If $\mu = r$ then $k = 1$, whence $n = 4$, $r = s = 2$ and $G = C_4$. Accordingly we suppose that $\mu \neq r$, and apply the results of Section 1 with $H = G - X = K_{1,s}$. We say that a vertex u in X is of type (a, b) if it has a neighbours of degree s in H and b neighbours of degree 1 in H. Thus a is 0 or 1, and $0 \leq b \leq s$.

We have $C = \begin{pmatrix} 0 & \mathbf{j}_s^\top \\ \mathbf{j}_s & O \end{pmatrix}$, and (cf. [13, Proposition 5.1.11])

$$\mu(\mu^2 - s)(\mu I - C)^{-1} = (\mu^2 - s)I + \mu C + C^2.$$

Since $a^2 = a$, Equations (2) and (4) become

$$\mu^2(\mu^2 - s) = a\mu^2 + 2\mu ab + b^2 + (\mu^2 - s)b, \tag{5}$$

$$-\mu(\mu^2 - s) = a\mu^2 + a\mu s + b\mu^2 + b\mu. \tag{6}$$

Since $(a, b) \neq (0, 0)$, Equations (5) and (6) yield just two possibilities:

$$a = 0,\ \ b = \mu^2 + \mu \neq 0,\ \ s = \mu(\mu^2 + 3\mu + 1) \ \text{ or } \ a = 1,\ \ \mu = -1,\ \ b \in \{1, s\}.$$

Thus if $\mu = -1$ then the central vertex of H is adjacent to all other vertices, and this contradicts the regularity of G since other vertices of H have degree less than n–1. It follows that $\mu \neq -1$ and the central vertex of H is adjacent to no vertices in X; in particular, $r = s = \mu(\mu^2 + 3\mu + 1)$. All vertices in X are of type $(0, \mu^2 + \mu)$, and counting in two ways the edges between X and H we have

$$|X|(\mu^2 + \mu) = \mu(\mu^2 + 3\mu + 1)(\mu^3 + 3\mu^2 + \mu - 1),$$

whence $|X| = (\mu^2 + 3\mu + 1)(\mu^2 + 2\mu - 1)$ and $n = |X| + s + 1 = (\mu^2 + 3\mu)^2$.

If we apply the compatibility condition (3) to vertices u, v of X, we find that

$$|\Delta_H(u) \cap \Delta_H(v)| = \begin{cases} 0 \text{ if } u \sim v \\ \mu \text{ if } u \not\sim v \end{cases}. \tag{7}$$

If X induces a clique then $|X| - 1 = r - \mu^2 - \mu$, whence

$$(\mu + 1)(\mu + 2)(\mu^2 + \mu - 1) = 0.$$

Therefore, either $\mu = -2$ and we have case (a), or $\mu = \frac{1}{2}(-1 \pm \sqrt{5})$ and we have case (b). If X does not induce a clique then it follows from (7) that $\mu \in I\!N$. In this situation, expressing r and k $(= |X|)$ in terms of μ, we find that $r - \mu^2 - \frac{k}{r}\mu^2 - 2\mu = 0$. By Lemma 2.1, G is strongly regular, and we have case (c) of the Theorem. □

In case (c) of Theorem 2.2, let $\mathcal{D} = \{\Delta_H(u) : u \in X\}$. If $\mu = 1$ then $\mathcal{D}$ consists of all 2-subsets of X, and so the star complement technique yields a unique graph G, necessarily the graph Cl_5 of Example 1. If $\mu = 2$ then it follows from (7) that $\mathcal{D}$ is a Steiner system $S(3, 6, 22)$. By a Theorem of Witt [42], there is only one such design, and so again G is unique. Here G is the Higman-Sims graph, the strongly regular graph HS with parameters $(100, 22, 0, 6)$ first constructed from $S(3, 6, 22)$ in [19]. Accordingly, we have:

Corollary 2.3 *Let G be a regular graph with $K_{1,s}$ $(s > 1)$ as a star complement for μ. If $\mu = 1$ then $G = Cl_5$, and if $\mu = 2$ then $G = C_4$ or HS.*

Note that conversely, if $\mu \in I\!N$ and if G is a strongly regular graph with parameters $((\mu^2 + 3\mu)^2, \mu(\mu^2 + 3\mu + 1), 0, \mu(\mu + 1))$ then G has as a star complement for μ the star induced by the closed neighbourhood of a vertex. Thus our proofs establish both the existence and uniqueness of strongly regular graphs with parameters $(16, 5, 0, 2)$ and $(100, 22, 0, 6)$. In the case $\mu = 3$, $\mathcal{D}$ would be a 3-(57,12,2) design, but it is shown in [24] that there is no such design; equivalently there is no strongly regular graph with parameters $(324, 57, 0, 12)$. Apparently, the cases $\mu > 3$ are open.

In the situation discussed in this section, the H-neighbourhoods form a design on a co-clique in H; generally the absence of structure makes it easier to deal with cases in which the H-neighbourhoods lie in a clique or co-clique. (However, there is no r-regular graph with K_t or $\overline{K_t}$ as a star complement for an eigenvalue $\mu \notin \{-1, 0, r\}$.) In Section 6 we shall encounter a situation in which less tractable H-neighbourhoods can be manipulated to create a design that can be exploited to establish a property of certain strongly regular graphs.

3 Windmills as star complements

Here we discuss the case in which G is r-regular with a windmill $K_1 \nabla h K_2$ ($h > 1$) as a star complement $H = G - X$ for an eigenvalue $\mu \neq r$. We write

$$M(a,b,c,d,e) = \left(\begin{array}{c|cc|cc|cc|c} d & e & e & e & e & e & e & \cdots \\ \hline e & a & b & c & c & c & c & \cdots \\ e & b & a & c & c & c & c & \cdots \\ \hline e & c & c & a & b & c & c & \cdots \\ e & c & c & b & a & c & c & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

Then H has adjacency matrix $C = M(0,1,0,0,1)$, with minimal polynomial $m(x) = (x^2-1)(x^2-x-2h) = x^4 - x^3 - (2h+1)x^2 + x + 2h$. Using [13, Proposition 5.1.11], we find that $m(\mu)(\mu I - C)^{-1} = M(a,b,c,d,e)$, where

$$a = \mu^3 - \mu^2 - (2h-1)\mu + 1,\ b = \mu^2 - 2h + 1,$$
$$c = \mu + 1,\quad d = (\mu-1)^2(\mu+1),\quad e = \mu^2 - 1.$$

The first row of $m(\mu)(\mu I - C)^{-1}$ has row-sum $-2h + 2\mu^2 h + \mu^3 - \mu^2 - \mu + 1$, and all other row-sums are $\mu^3 + \mu^2 - \mu - 1$. Note that, for any $u \in X$, $m(\mu)\langle \mathbf{b}_u, \mathbf{j}\rangle$ is the sum of row-sums indexed by $\Delta_H(u)$. Now suppose that u has f neighbours of degree 2 in H, while w is the central vertex of H. From Equation (4), we find that

$$f = 1 - \mu, \quad \text{if } u \sim w, \tag{8}$$

and

$$f = \frac{2h-2}{\mu+1} - \mu + 2, \quad \text{if } u \not\sim w. \tag{9}$$

We consider the case $\mu > 0$. Since also $\mu \neq 1$ (because H has 1 as an eigenvalue), this eliminates the first possibility (8). Hence $r = 2h$ and Equation (2) yields

$$\mu(\mu^2-1)(\mu^2-\mu-r) = \mu(\mu^2-\mu-r)f + (\mu+1)f^2 + 2(\mu^2-\mu-r)b, \tag{10}$$

where b is the number of triangles $uiju$ with i and j neighbours of w (see [34, Equation (9)]). In (10), we substitute for f from (9) to obtain

$$r = \mu^2(\mu+3) - 2\tfrac{\mu+1}{\mu-1}b. \tag{11}$$

Since also $kf = r(r-2)$ and $f = \mu(\mu+1) - \frac{2b}{\mu-1}$, we find that

$$r - \mu^2 - \tfrac{k}{r}\mu^2 - 2\mu = \frac{-2b\{\mu^2 - 2\mu + (\mu+1)f)\}}{(\mu-1)f}. \tag{12}$$

Note that $\mu^2 - 2\mu + (\mu+1)f > 0$. We show that also $\mu - 1 > 0$; this is immediate if $\mu \in I\!N$. If $\mu \notin I\!N$ then $r \geq k$ (because some algebraic conjugate of μ has multiplicity k). Then $f \geq r-2$. If $f = r-2$ then $k = r$ and G has just three eigenvalues; thus G is strongly regular. By Lemma 2.1 and Equation (12) we have $b = 0$, whence $f = \mu(\mu+1)$ and $r = 2\mu(\mu+1)$: then $(\mu+2)(\mu-1) = 0$, a contradiction. If $f = r-1$ then $k(r-1) = r(r-2)$, whence $r = 2$ and $k = 0$, a contradiction. If $f = r$ then $k = 1$ by Proposition 1.3(ii), and so $r = 3$, a contradiction.

Now it follows from (12) and Lemma 2.1 that $b = 0$ and G is strongly regular. In this situation, we have $r = \mu^2(\mu+3)$, $f = \mu(\mu+1)$ and $k = \mu(\mu+3)(\mu^2+2\mu-2)$. We can now extend Theorem 3.2 of [34] as follows:

Theorem 3.1 *If the r-regular graph G has $K_1 \nabla hK_2$ ($h > 1$) as a star complement for the positive eigenvalue μ then $r = 2h$ and G is strongly regular with parameters*

$$((\mu^2+3\mu-1)^2, \mu^2(\mu+3), 1, \mu(\mu+1)).$$

Conversely, if G is a strongly regular graph with parameters $((\mu^2+3\mu-1)^2, \mu^2(\mu+3), 1, \mu(\mu+1))$ ($\mu > 0$) and $h = \frac{1}{2}\mu^2(\mu+3)$ then G has $K_1 \nabla hK_2$ as a star complement for the eigenvalue μ. Examples are known to arise when $\mu = 2$ and $\mu = 4$: for $\mu = 2$, G is the unique strongly regular graph with parameters $(81, 20, 1, 6)$ [3, 41], and when $\mu = 4$, a rank 3 graph with parameters $(729, 112, 1, 20)$ can be constructed from a projective ternary code [20]. The case $\mu = 3$ remains open.

The uniqueness of a strongly regular graph with parameters $(81, 20, 1, 6)$, previously established in [3], is proved in [41] using the star complement technique. Indeed the following suffices to establish existence and uniqueness. For distinct vertices u, v of X, let $|\Delta_H(u) \cap \Delta_H(v)| = \alpha(u, v)$, and let $\beta(u, v)$ be the number of edges ij in H with i adjacent to u and j adjacent to v. Then the compatibility condition yields:

$$\alpha(u,v)\mu + \beta(u,v) = \begin{cases} \mu+1 & \text{if } u \sim v \\ \mu(\mu+1) & \text{if } u \not\sim v. \end{cases} \tag{13}$$

When $\mu = 2$ it transpires that there is essentially just one way to add to H the k vertices of X in such a way that (13) is satisfied for all $u, v \in X$.

4 Generalizations

In Theorems 2.2 and 3.1, our arguments led to the situation in which the regular graph G has a star complement $H = G - X$ induced by $\Delta^*(w)$, where $\Delta^*(w)$ is the closed neighbourhood of some vertex w of G, and the neighbourhood $\Delta(w)$ induces a regular subgraph. In general this is a favourable situation because the following result ensures that the H-neighbourhoods of vertices in X all have the same size:

Proposition 4.1 [34, Lemma 3.1] *Let G be an r-regular graph ($r > 0$) with an eigenvalue μ of multiplicity k. Suppose that G has a star set X such that $\overline{X} = \Delta^*(w)$ and $\Delta(w)$ induces an e-regular graph. Then*
(i) *$\mu \neq -1$,*
(ii) *if $\mu = r$ then G is a cocktail-party graph,*
(iii) *if $\mu \neq r$ then X induces a d-regular graph, where*

$$d = \frac{\mu(r - e + \mu)}{\mu + 1}.$$

This can be proved using the relation

$$\mu P\mathbf{e}_i = AP\mathbf{e}_i = PA\mathbf{e}_i = \sum_{j\in\Delta(i)} P\mathbf{e}_j \quad (i \in V(G))$$

together with linear independence of the vectors $P\mathbf{e}_u$ ($u \in X$). In similar fashion, one can prove that in a regular graph, a uniform star set induces a regular subgraph: if G is r-regular, and $G - X$ is regular of degree $r - c$ then X induces a regular subgraph of degree $\mu + c$ [33, Theorem 2.1]. Here, $X \dot\cup \overline{X}$ is an equitable bipartition of $V(G)$ (cf. [6, 7]). In the situation of Proposition 4.1, $\{w\} \cup \Delta(w) \cup \overline{\Delta^*(w)}$ is an equitable partition of $V(G)$ with divisor matrix

$$D = \begin{pmatrix} 0 & r & 0 \\ 1 & e & r-e-1 \\ 0 & r-d & d \end{pmatrix}.$$

The characteristic polynomial of D is $(x - r)(x^2 + (r - d - e)x - d)$. When $0 < \mu < r$, we have $\mu^2 + (r - d - e)\mu - d = 0$ by Proposition 4.1(iii), and so the eigenvalues of D are r, μ and $-d/\mu$. If G is strongly regular then clearly we have tight interlacing of these eigenvalues with those of G, in the sense of [18, Section 2]. In the reverse direction, we have:

Proposition 4.2 *In the situation of Proposition 4.1, let f be the number of vertices in each neighbourhood $\Delta_H(u)$ ($u \in X$). Suppose that $0 < \mu < r$ and we have tight interlacing of the eigenvalues of D with those of G. Then $f \geq \mu(\mu + 1)$, with equality if and only if G is strongly regular.*

Proof. Let the eigenvalues of G other than μ and r be $\theta_1, \ldots, \theta_r$ in non-decreasing order. In the case of tight interlacing, we have $\theta_1 = -d/\mu$, and so

$$0 = \sum_{i=1}^{r} \theta_i + k\mu + r \geq -\frac{rd}{\mu} + k\mu + r = k\mu - \frac{r(r-e-1)}{\mu+1} = \frac{k\{\mu(\mu+1) - f\}}{\mu+1}.$$

Hence $f \geq \mu(\mu + 1)$, and equality holds if and only if $\theta_1 = \cdots = \theta_r$ (that is, if and only if G has just three distinct eigenvalues). $\square$

Now we turn to other star complements H which have received attention (not necessarily in full generality) in the context of regular graphs. These include $K_{r,s}$ ($r \geq s > 1$) (see [23]), $K_{1,r}^{(s)}$ ($r > 1, s \geq 1$) (see [37]),

$K_1\nabla hK_q$ ($q>2$) (see [34]) and $K_{1,r}\,\dot\cup\,qK_1$ (see [22, 35]). Here $K_{1,r}^{(s)}$ denotes the graph of order $1+r(s+1)$ obtained from $K_{1,r}$ by adding s pendant edges at each vertex of degree 1.

As an illustration of the first type, we have:

Theorem 4.3 [23, Theorem 3.1] *The graph Sch_{10} is the unique maximal graph with $K_{2,5}$ as a star complement for a multiple eigenvalue other than -1.*

As an example of the second type, we have the following characterization of the Hoffman-Singleton graph HoS; this is the unique strongly regular graph with parameters $(50,7,0,1)$, otherwise known as the Moore graph of degree 7 and diameter 2 (see [21]).

Theorem 4.4 [37, Theorem 2.3] *If G is a regular graph with $K_{1,7}^{(2)}$ as a star complement for the eigenvalue 2 then $G=HoS$.*

Here the compatibility condition is used to prove that G has girth 5. We note in passing that that $K_{1,7}^{(2)}$ is the (3-harmonic) Grünewald tree [17] of order 22. It arises in HoS by adding one vertex neighbourhood to a maximal independent set (of size 15). If there exists a Moore graph G of degree 57 and diameter 2 then it has at most 400 independent vertices; moreover if there are 400 such vertices, one vertex neighbourhood can be added to obtain an (8-harmonic) Grünewald tree T of order 457 (see [16, pp. 99-100]). Since T is an induced subgraph without 7 as an eigenvalue, it extends to a star complement H for 7 (of order 1521). If such a graph G exists, in principle it can be constructed by identifying H and 1729 H-neighbourhoods.

When $H=K_1\nabla hK_q$ ($q>2$), the arguments of Section 3 may be used to prove the following result, which extends [34, Theorem 3.2].

Theorem 4.5 *Let G be an r-regular graph with $K_1\nabla hK_q$ $(h>1, q>1)$ as a star complement for the eigenvalue μ, where $r=hq$ and $\mu>q-1$. Then G is strongly regular with parameters*

$$((\mu^2+3\mu-1)^2, \mu^2(\mu+3), 1, \mu(\mu+1)).$$

Here the assumption $r=hq$ avoids the calculation of $\langle \mathbf{b}_u, \mathbf{j}\rangle$ ($u\in X$), since Proposition 4.1 yields

$$f=\frac{r+(q-1)\mu-\mu^2}{\mu+1}.$$

Then, in place of (12), we have

$$r-\mu^2-\tfrac{k}{r}\mu^2-2\mu=\frac{-2b}{f}\left(\mu+\frac{(f-1)\mu+f}{\mu-q+1}\right),$$

and we can invoke Lemma 2.1. An example with $\mu=3, q=3$ is described in [34, Section 3].

Some star complements of the fourth type (consisting of a star and isolated vertices) are discussed in Section 6, in the context of maximal independent sets. The motivation lies with eigenvalues of maximal multiplicity, considered in the next section.

5 The multiplicity of an eigenvalue

Let G be a graph (not necessarily regular) of order n with μ as an eigenvalue of multiplicity k, and let $t = n - k$. In this section we discuss upper bounds for n, and hence for k, as functions of t. We noted in Section 1 that if $\mu \neq -1, 0$ then $n < t + 2^t$. We shall see that this bound can be improved to a sharp upper bound which is a quadratic function of t. A further improvement can be made when G is regular, and in this case we find that the graphs which attain the bound are strongly regular. The results here are taken from [2], and the arguments refine those of [31].

In the notation of Theorem 1.2, let S be the $t \times n$ matrix $(B|C - \mu I)$, with columns $\mathbf{s}_u$ $(u = 1, \ldots, n)$. Then

$$\mu I - A = S^\top (\mu I - C)^{-1} S, \tag{14}$$

and we have, for all vertices u, v of G,

$$\langle \mathbf{s}_u, \mathbf{s}_v \rangle = \begin{cases} \mu & \text{if } u = v \\ -1 & \text{if } u \sim v \\ 0 & \text{otherwise} \end{cases} .$$

We now define functions $F_1, \ldots, F_n$ from $I\!R^t \to I\!R$ as follows:

$$F_u(\mathbf{x}) = \langle \mathbf{s}_u, \mathbf{x} \rangle^2 \qquad (\mathbf{x} \in I\!R^t).$$

It can be shown that if, additionally, μ is different from the largest eigenvalue λ_1 then these n functions are linearly independent. The proof makes use of the pairwise orthogonality of the subspaces $\mathcal{E}(\lambda_1)$, $\mathcal{E}(\mu)$, $\mathcal{E}(-\mu^2)$ (the last of which is the zero subspace if $-\mu^2$ is not an eigenvalue of G). Now the space of homogeneous quadratic functions on $I\!R^t$ has dimension $\frac{1}{2}t(t+1)$ and so $n \leq \frac{1}{2}t(t+1)$. The case $\mu = \lambda_1$ is treated separately to obtain the following result.

Theorem 5.1 [2, Theorem 2.3] *Let G be a graph of order n, let μ be an eigenvalue of G, and let t be the codimension of $\mathcal{E}(\mu)$. If $\mu \notin \{-1, 0\}$, then either* (a) $n \leq \frac{1}{2}t(t+1)$ *or* (b) $\mu = 1$ *and* $G = K_2$ *or* $2K_2$.

Example 2 The bound of Theorem 5.1(a) is attained when G is the graph obtained from the regular graph $L(K_9)$ by switching with respect to a clique of order 8; here $\mu = -2$, $t = 8$ and $n = 36$. □

In the case that μ is a non-main eigenvalue and $t > 2$ we can reduce our upper bound by 1 as follows. We define an additional function F by

$$F(\mathbf{x}) = \langle \mathbf{j}, \mathbf{x} \rangle^2 \qquad (\mathbf{x} \in I\!R^t),$$

and then show that the functions $F, F_1, F_2, \ldots, F_n$ are linearly independent. Thus $n + 1 \leq \frac{1}{2}t(t+1)$, and we have:

Theorem 5.2 [2] *Let μ be an eigenvalue of an r-regular graph of order n, and let t be the codimension of $\mathcal{E}(\mu)$. If $\mu \notin \{-1, 0, r\}$ and $t > 2$ then*

$$n \leq \tfrac{1}{2}t(t+1) - 1 = \tfrac{1}{2}(t-1)(t+2).$$

The bound of Theorem 5.2 is also sharp; in fact, we have the following result, for which we give a proof considerably shorter than the original. Recall that if G is strongly regular of order n, with eigenvalues r, μ', μ of multiplicities $1, k', k$ $(1 < k' \leq k)$ then $n \leq \frac{1}{2}k'(k'+3)$ (the 'absolute bound', see [39]); and G is said to be *extremal* if $n = \frac{1}{2}k'(k'+3)$. Note that if G is an extremal strongly regular graph then so is $\overline{G}$.

Theorem 5.3 [2] *The regular graphs attaining the bound of Theorem 5.2 are precisely the extremal strongly regular graphs.*

Proof. First, let G be an extremal strongly regular graph with eigenvalues r, μ', μ of multiplicities $1, k', k$, where $1 < k' \leq k$. Thus if G has n vertices then $n = \frac{1}{2}k'(k'+3)$. If $t = n - k$ then $k' = t - 1$ and so $n = \frac{1}{2}(t-1)(t+2)$ as required.

Conversely, if G is a regular graph that attains the bound of Theorem 5.2, then every homogeneous quadratic function on $I\!R^t$ is a linear combination of $F_1, F_2, \ldots F_n$ and F. In particular,

$$\langle \mathbf{x}, \mathbf{x} \rangle = \sum_{u=1}^{n} \epsilon_u F_u(\mathbf{x}) + \gamma F(\mathbf{x}), \tag{15}$$

for some scalars $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ and γ. It follows that

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{u=1}^{n} \epsilon_u \langle \mathbf{s}_u, \mathbf{x} \rangle \langle \mathbf{s}_u, \mathbf{y} \rangle + \gamma \langle \mathbf{j}, \mathbf{x} \rangle \langle \mathbf{j}, \mathbf{y} \rangle. \tag{16}$$

Let $\mathbf{e} = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)^\top$. Taking $\mathbf{x} = \mathbf{s}_i,\ \mathbf{y} = -\mathbf{j}$ $(i = 1, \ldots, n)$ in (16), we find that

$$(\mu I - A)\ \mathbf{e} = (1 - \gamma \langle \mathbf{j}, \mathbf{j} \rangle)\ \mathbf{j}. \tag{17}$$

Taking $\mathbf{x} = \mathbf{s}_i$ in (15), we find that

$$(\mu^2 I + A)\ \mathbf{e} = (\mu - \gamma)\ \mathbf{j}. \tag{18}$$

From (17) and (18) we see that $(\mu^2 + \mu)\mathbf{e}$ is a scalar multiple of $\mathbf{j}$. Since $\mu^2 + \mu \neq 0$, $\mathbf{e} = \epsilon\ \mathbf{j}$ for some ϵ. Now, taking $\mathbf{x} = \mathbf{s}_i, \mathbf{y} = \mathbf{s}_j$ $(i \neq j)$ in (16), we have

$$\langle \mathbf{s}_i, \mathbf{s}_j \rangle = \epsilon \sum_{i=1}^{n} \langle \mathbf{s}_u, \mathbf{s}_i \rangle \langle \mathbf{s}_u, \mathbf{s}_j \rangle + \gamma.$$

It follows that if $i \not\sim j$ then $0 = \epsilon a_{ij}^{(2)} + \gamma$. Since G is not complete, we deduce that $\epsilon \neq 0$, and $a_{ij}^{(2)} = -\epsilon^{-1}\gamma$ when $i \not\sim j$. Similarly, if $i \sim j$ then $a_{ij}^{(2)} = 2\mu - \epsilon^{-1}(\gamma + 1)$, and this completes the proof. □

The strongly regular graphs that arise in Theorem 5.3 have parameters (n, r, e, f), where (see [2])

$$e = \mu^2 + r - \frac{n\mu^2(\mu+1)}{n+2\mu-2r}, \quad f = \mu^2 + 2\mu + r - \frac{n\mu(\mu+1)^2}{n+2\mu-2r}.$$

The known extremal strongly regular graphs are $C_5, Sch_{10}, Sch_{16}, McL_{112}$ and McL_{162}.

6 Extremal strongly regular graphs

In investigating extremal strongly regular graphs of degree r, we may assume (by passing to the complement if necessary) that the eigenvalue $\mu \neq r$ of larger multiplicity k is positive. If G is such a graph with n vertices, and if $t = n - k$, then G has $n - t + 1$ positive eigenvalues; then (by interlacing) a co-clique in G has size at most $t - 1$. This bound is attained in all three known examples, namely C_5, Sch_{10} and McL_{112}. Here we describe how star complements are used to prove the converse: if G has $t - 1$ independent vertices then G is one of these three graphs.

Cameron, Goethals and Seidel [4] have shown that the various parameters of an extremal strongly regular graph G are again polynomial functions of μ. These functions are as follows, where λ is the third eigenvalue of G (see [5, Chapter 8] and [2]):

$$\begin{aligned}
n &= (2\mu+1)^2(2\mu^2+2\mu-1), \\
r &= 2\mu^3(2\mu+3), \\
e &= \mu(2\mu-1)(\mu^2+\mu-1), \\
f &= \mu^3(2\mu+3), \\
\lambda &= -\mu^2(2\mu+3), \\
k &= 2\mu(\mu+1)(2\mu-1)(2\mu+3), \\
t &= 4\mu^2+4\mu-1.
\end{aligned}$$

Moreover, either $\mu \in I\!N$ or $G = C_5$ and $\mu^2 + \mu = 1$.

We assume here that μ is an integer greater than 1, for otherwise G is C_5 or Sch_{10}. By interlacing, the largest eigenvalue of an induced subgraph of order t is at least μ, and so G has no induced subgraph $K_2 \dot\cup (t-2)K_1$. Now suppose that G has a co-clique C of order $t - 1$ $(= 4\mu^2 + 4\mu - 2)$. Then each vertex v outside C is adjacent to at least 2 vertices of C; in other words, $C + v$ has the form $K_{1,s} \dot\cup (t-s-1)K_1$ $(2 \leq s \leq t-1)$. The vertex v can be chosen such that $s \neq \mu^2$ for otherwise, counting in two ways the number of edges with a vertex in C, we have $r(t-1) = \mu^2(n-t+1)$, whence $4\mu(2\mu+3) = (2\mu+1)^2 - 2$, a contradiction. Accordingly v may be chosen such that $C + v$ is a star complement H for μ.

The neighbourhoods $\Delta_H(u)$ $(u \in X)$ do not themselves form a design, but they can be manipulated to construct a design as follows (see [35]). Equations (2) and (4) show that, up to isomorphism, there are at most four different graphs $H + u$ ($u \in X$). If X is partitioned accordingly as $X_1 \dot\cup X_2 \dot\cup X_3 \dot\cup X_4$, then (with an appropriate choice for X_1, X_2), G^* is defined as the graph obtained from $\overline{G}$ by adding an isolated vertex x and switching with respect to $\{x\} \dot\cup X_1 \dot\cup X_2$. The subgraph K of G^* induced by $\{x\} \dot\cup C$ is complete of order t, and it turns out that the $\binom{t}{2}$ neighbourhoods $\Delta_K(y)$ ($y \in V(G^*) \setminus V(K)$) form a tight 4-design $\mathcal{D}$, in the sense of [5, Chapter 1]. Such designs are rare; indeed if $4 \leq t - s < t - 2$ then $\mathcal{D}$ is necessarily the Steiner system $S(4, 7, 23)$ (see [5, Theorems 1.52 and 1.54]). In these circumstances, $\mu = 2$ and we can reverse the construction of G^* from G to obtain McL_{112}.

The cases $s = 2$, $s = t - 3$, $s = t - 2$ and $s = t - 1$ can be eliminated using the equation $\langle \mathbf{j}, \mathbf{j} \rangle = n/(\mu - r)$, which follows from the observation that $\langle (r - \mu)\mathbf{j}, \mathbf{j} \rangle = \sum_{i=1}^{n} \langle \mathbf{s}_i, \mathbf{j} \rangle$. The outcome can be summarized as follows.

Theorem 6.1 [2] *Let G be an extremal strongly regular graph in which an eigenvalue μ of largest multiplicity is a positive integer. If a star complement for μ has the form $K_{1,s} \dot\cup (t - s - 1)K_1$ $(2 \leq s \leq t - 1)$ then either*
(a) $\mu = 1$, $t = 7$, $s \in \{2, 5\}$ *and* $G = Sch_{10}$, *or*
(b) $\mu = 2$, $t = 23$, $s = 16$ *and* $G = McL_{112}$.

Corollary 6.2 [2] *Let G be an extremal strongly regular graph in which an eigenvalue μ of largest multiplicity is positive. Then the independence number of G is at most $4\mu^2 + 4\mu - 2$, with equality if and only if G is C_5, Sch_{10} or McL_{112}.*

When the eigenvalue μ of largest multiplicity is negative, an analogous result holds, with ‘clique number’ in place of ‘independence number’ and Sch_{16}, McL_{162} in place of Sch_{10}, McL_{112}.

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