Rowlinson P (2014) Eigenvalue multiplicity in cubic graphs, *Linear Algebra and Its Applications*, 444, pp. 211-218.

EIGENVALUE MULTIPLICITY IN CUBIC GRAPHS

Peter Rowlinson[1]
Mathematics and Statistics Group
Institute of Computing Science and Mathematics
University of Stirling
Scotland FK9 4LA

Abstract

Let G be a connected cubic graph of order n with μ as an eigenvalue of multiplicity k. We show that (i) if $\mu \notin \{-1, 0\}$ then $k \leq \frac{1}{2}n$, with equality if and only if $\mu = 1$ and G is the Petersen graph; (ii) If $\mu = -1$ then $k \leq \frac{1}{2}n + 1$, with equality if and only if $G = K_4$; (iii) If $\mu = 0$ then $k \leq \frac{1}{2}n + 1$, with equality if and only if $G = \overline{2K_3}$.

AMS Classification: 05C50

Keywords: cubic graph, eigenvalue, star complement.

[1]Tel.:+44 1786 467468; fax +44 1786 464551; email: p.rowlinson@stirling.ac.uk

1 Introduction

Let G be a regular graph of order n with μ as an eigenvalue of multiplicity k, and let $t = n - k$. Thus the corresponding eigenspace $\mathcal{E}(\mu)$ of a $(0,1)$-adjacency matrix A of G has dimension k and codimension t. From [1, Theorem 3.1], we know that if $\mu \notin \{-1, 0\}$ and $t > 2$ then $k \leq n - \frac{1}{2}(-1 + \sqrt{8n+9})$, equivalently $k \leq \frac{1}{2}(t+1)(t-2)$. For cubic graphs, this quadratic bound improves an earlier cubic bound noted in [4, p.162]. In fact, when $\mu \neq 0$ and G is connected, a linear bound follows easily from the equation $\mathrm{tr}(A) = 0$. To see this, note first that if $k \geq \frac{1}{2}n$ then μ is an integer, for otherwise it has an algebraic conjugate which is a second eigenvalue of multiplicity $\frac{1}{2}n$. It follows that if G is a connected cubic graph then $\mu \in \{-2, -1, 0, 1, 2\}$ (see [3, Sections 1.3 and 3.2]). If $k = n - 1$ then G is complete, $n = 4$ and $\mu = -1$; otherwise let d be the mean of the eigenvalues other than 3 and μ, so that $3 + k\mu + (n-k-1)d = 0$. We have $-3 \leq d < 3$; moreover, if $d = -3$ then G is bipartite, $k = n - 2$ and $\mu = 0$ (see [3, Theorems 3.2.3 and 3.2.4]). We deduce:

(a) if $\mu = -2$ then $k < \frac{3}{5}n$, i.e. $k < \frac{3}{2}t$;
(b) if $\mu = -1$ then $k \leq \frac{3}{4}n$, i.e. $k \leq 3t$;
(c) if $\mu = 0$ then $k \leq n - 2$;
(d) if $\mu = 1$ then $k < \frac{3}{4}n - \frac{3}{2}$, i.e. $k < 3t - 6$;
(e) if $\mu = 2$ then $k < \frac{3}{5}n - \frac{6}{5}$, i.e. $k < \frac{3}{2}t - 3$.

We use star complements to improve these bounds, and to determine all the graphs for which the new bounds are attained. Our main result is the following; here and throughout we use the notation of the monograph [3].

Theorem 1.1. *Let G be a connected cubic graph of order n with μ as an eigenvalue of multiplicity k.*
(i) *If $\mu \notin \{-1, 0\}$ then $k \leq \frac{1}{2}n$, with equality if and only if $\mu = 1$ and G is the Petersen graph.*
(ii) *If $\mu = -1$ then $k \leq \frac{1}{2}n + 1$, with equality if and only if $G = K_4$.*
(iii) *If $\mu = 0$ then $k \leq \frac{1}{2}n + 1$, with equality if and only if $G = \overline{2K_3}$.*

It follows that if G is a connected cubic graph of order $n > 10$ with μ as an eigenvalue of multiplicity k then $k \leq \frac{1}{2}n - 1$ when $\mu \notin \{-1, 0\}$, and $k \leq \frac{1}{2}n$ otherwise.

2 Preliminaries

Let G be a graph of order n with μ as an eigenvalue of multiplicity k. A *star set* for μ in G is a subset X of the vertex-set $V(G)$ such that $|X| = k$ and the induced subgraph $G - X$ does not have μ as an eigenvalue. In this situation, $G - X$ is called a *star complement* for μ in G. The fundamental properties of star sets and star complements are established in [3, Chapter 5]. We shall require the following results, where for any $X \subseteq V(G)$, we write G_X for the subgraph of G induced by X. We take $V(G) = \{1, \ldots, n\}$, and write $u \sim v$ to mean that vertices u and v are adjacent.

Theorem 2.1. (See [3, Theorem 5.1.7].) *Let X be a set of k vertices in G and suppose that G has adjacency matrix $\begin{pmatrix} A_X & B^\top \\ B & C \end{pmatrix}$, where A_X is the adjacency matrix of G_X.*
(i) *Then X is a star set for μ in G if and only if μ is not an eigenvalue of C and*

$$\mu I - A_X = B^\top(\mu I - C)^{-1}B. \tag{1}$$

(ii) *If X is a star set for μ then $\mathcal{E}(\mu)$ consists of the vectors $\begin{pmatrix} \mathbf{x} \\ (\mu I - C)^{-1}B\mathbf{x} \end{pmatrix}$ $(\mathbf{x} \in I\!R^k)$.*

Let $H = G - X$, where X is a star set for μ. The columns $\mathbf{b}_u$ $(u \in X)$ of B are the characteristic vectors of the H-neighbourhoods $\Delta_H(u) = \{v \in V(H) : u \sim v\}$ $(u \in X)$. Eq. (1) shows that

$$\mathbf{b}_u^\top(\mu I - C)^{-1}\mathbf{b}_v = \begin{cases} \mu \text{ if } u = v \\ -1 \text{ if } u \sim v \\ 0 \text{ otherwise,} \end{cases}$$

and we deduce from Theorem 2.1:

Lemma 2.2. *If X is a star set for μ, and $\mu \notin \{-1, 0\}$, then the neighbourhoods $\Delta_H(u)$ $(u \in X)$ are non-empty and distinct.*

Let P be the matrix of the orthogonal projection of $I\!R^n$ onto $\mathcal{E}(\mu)$ with respect to the standard orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ of $I\!R^n$. Since P is a polynomial in A [3, Equation 1.5] we have $\mu P\mathbf{e}_i = AP\mathbf{e}_i = PA\mathbf{e}_i$ $(i = 1, \ldots, n)$, whence:

Lemma 2.3. $\mu P\mathbf{e_i} = \sum_{j\sim i} P\mathbf{e}_j$ $(i = 1, \ldots, n)$.

The next observation follows from [3, Proposition 5.1.1].

Lemma 2.4. *The subset S of $V(G)$ lies in a star set for μ if and only if the vectors $P\mathbf{e}_i$ $(i \in S)$ are linearly independent.*

By interlacing [3, Corollary 1.3.12] we have:

Lemma 2.5. *If S is a star set for μ in G and if U is a proper subset of S then $S \setminus U$ is a star set for μ in $G - U$.*

We shall also require:

Lemma 2.6. (See [3, Theorem 5.1.6].) *Let μ be an eigenvalue of the graph G. If G is connected then G has a connected star complement for μ.*

In the case of connected cubic graphs, we can therefore make use of the following result.

Proposition 2..7. *Let G be a connected cubic graph of order n with μ as an eigenvalue of multiplicity $k \geq \frac{1}{2}n$. Let H be a connected star complement for μ, and let $H = G - X$, $\overline{X} = V(H)$, $|\overline{X}| = t$. Then each vertex in X is adjacent to some vertex in $\overline{X}$, and one of the following holds:*

(a) $k = t$, $|E(X, \overline{X})| = t$ *and* H *is unicyclic,*
(b) $k = t$, $|E(X, \overline{X})| = t + 2$ *and* H *is a tree,*
(c) $k = t + 2$, $|E(X, \overline{X})| = t + 2$, $\mu \in \{-1, 0\}$ *and* H *is a tree.*

Proof. If $u \in X$ then $\mu P\mathbf{e}_u = \Sigma_{i \in \Delta_X(u)} P\mathbf{e}_i + \Sigma_{j \in \Delta_H(u)} P\mathbf{e}_j$, where $\Delta_X(u) = \{i \in X : i \sim u\}$. It now follows from Lemma 2.4 that $\Delta_H(u) \neq \emptyset$. For $j \in \overline{X}$, let $d_j = |\Delta_H(j)|$, $e_j = |\Delta_X(j)|$. Then

$$|E(X, \overline{X})| = \Sigma_{j \in \overline{X}} e_j = 3t - \Sigma_{j \in \overline{X}} d_j = 3t - 2|E(H)|.$$

Since $|E(H)| \geq t - 1$ we deduce that $|E(X, \overline{X})| \leq t + 2$. Since $k \geq \frac{1}{2}n$ and each vertex in X has a neighbour in $\overline{X}$, we have

$$t \leq k \leq |E(X, \overline{X})| \leq t + 2 \quad \text{and} \quad |E(H)| \leq t.$$

If $|E(H)| = t$ then H is unicyclic and $t = k = |E(X, \overline{X})|$: this is case (a) of the Proposition. If $|E(H)| = t - 1$ then H is a tree and $|E(X, \overline{X})| = t + 2$; moreover, k is t or $t + 2$ because n is even. If $k = t$ we have case (b). If $k = t + 2$ then $|\Delta_H(i)| = 1$ for each $i \in X$ and so there are two vertices in X with a common H-neighbourhood. We deduce from Lemma 2.2 that $\mu \in \{-1, 0\}$ and so we have case (c). $\square$

It follows that $k \leq \frac{1}{2}n$ when $\mu \notin \{-1, 0\}$, and $k \leq \frac{1}{2}n + 1$ when $\mu \in \{-1, 0\}$. In Sections 3 and 4 we determine the graphs in which these bounds are attained. It is clear from Proposition 2.7 that the edges between X and $\overline{X}$ play a crucial role. The authors of [2] have determined all the graphs for which $E(X, \overline{X})$ is a perfect matching, equivalently all the graphs for which $B = I$ in Eq.(1). Their result is the following.

Theorem 2.8. *Let* G *be a graph with* X *as a star set for the eigenvalue* μ*. If* $E(X, \overline{X})$ *is a perfect matching then one of the following holds:* (a) $G = K_2$ *and* $\mu = \pm 1$, (b) $G = C_4$ *and* $\mu = 0$, (c) G *is the Petersen graph and* $\mu = 1$.

We shall see that when $E(X, \overline{X})$ is not a perfect matching, and G is a connected cubic graph with $k \geq \frac{1}{2}n$, it suffices to consider a limited number of configurations from which we can construct a fragment of G. In most cases, we invoke Lemmas 2.3 and 2.4 to obtain a contradiction. In the remaining cases, either the fragment is G itself or we derive a contradiction from Theorem 2.1(ii). The configurations that we consider when $\mu \notin \{-1, 0\}$ are illustrated in Fig. 1, labelled in accordance with various subcases described in Section 3.

3 The case $\mu \notin \{-1, 0\}$

We retain the notation of Section 2. We assume that G is a connected cubic graph, with $\mu \notin \{-1, 0\}$ and $k = \frac{1}{2}n$. Thus $\mu \in \{-2, 1, 2\}$. By Lemma 2.6, we know that G has a connected star complement H for μ; accordingly we have to deal with cases (a) and (b) of Proposition 2.7. In case (a), the t edges in $E(X, \overline{X})$ form a perfect matching (and H is a cycle) because the vertices in X have distinct H-neighbourhoods. Thus $\mu = 1$ and G is the Petersen graph, by Theorem 2.8. For the remainder of this section, we therefore assume that $|E(X, \overline{X})| = t + 2$ and H is a tree.

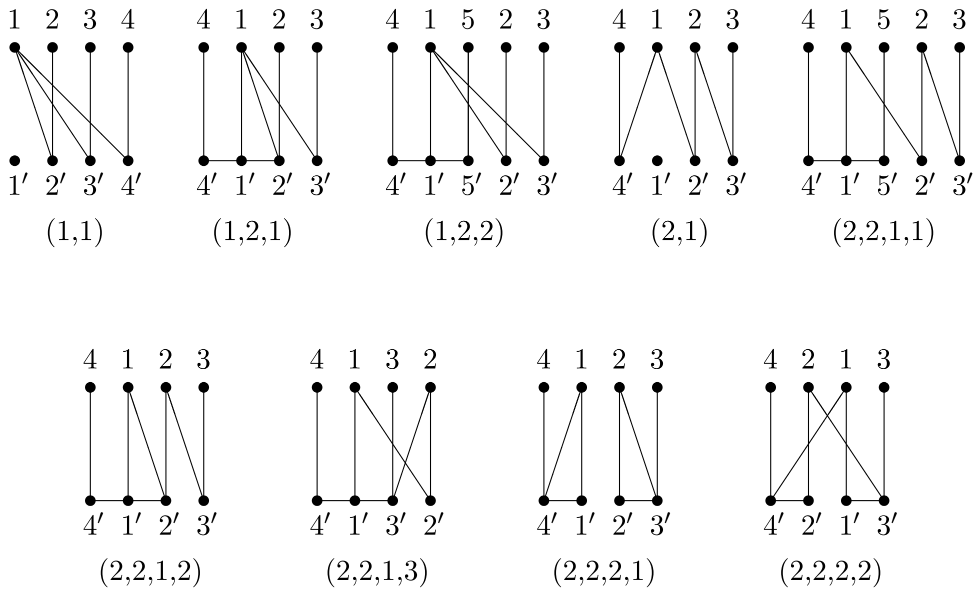

Figure 1: Configurations in the case $\mu \notin \{-1, 0\}$

We take $X = \{1, 2, \dots, t\}$, $\overline{X} = \{1', 2', \dots, t'\}$, and for each $i \in X$ we denote $\Sigma\{P\mathbf{e}_h : h \in \Delta_X(i)\}$ by $\mathbf{v}_i$. We distinguish two cases: (1) X contains a vertex adjacent to three vertices of H, (2) X contains two vertices with H-neighbourhoods of size 2. In case (1), we may take $|\Delta_H(1)| = 3$ and $\Delta_H(i) = \{i'\}$ $(i = 2, \dots, t)$. There are two subcases: without loss of generality, either (1,1) $\Delta_H(1) = \{2', 3', 4'\}$ or (1,2) $\Delta_H(1) = \{1', 2', 3'\}$. In subcase (1,1), we have

$$\mu P\mathbf{e}_1 = P\mathbf{e}_{2'} + P\mathbf{e}_{3'} + P\mathbf{e}_{4'} = \mu P\mathbf{e}_2 - \mathbf{v}_2 + \mu P\mathbf{e}_3 - \mathbf{v}_3 + \mu P\mathbf{e}_4 - \mathbf{v}_4.$$

For $\mu = -2, 1, 2$ respectively we obtain :

$$2P\mathbf{e}_1 = 2P\mathbf{e}_2 + \mathbf{v}_2 + 2\mathbf{e}_3 + \mathbf{v}_3 + 2P\mathbf{e}_4 + \mathbf{v}_4,$$

$$P\mathbf{e}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = P\mathbf{e}_2 + P\mathbf{e}_3 + P\mathbf{v}_4,$$

$$2P\mathbf{e}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = 2P\mathbf{e}_2 + 2P\mathbf{e}_3 + 2P\mathbf{v}_4.$$

In each case, the imbalance of summands of the form $P\mathbf{e}_i$ $(i \in X)$ yields a contradiction to Lemma 2.4.

In subcase (1,2), H has degree sequence $1^{(2)}, 2^{(t-2)}$ and so H is a path; its endvertices are $2'$ and $3'$. Note that $t > 3$ because $2 \not\sim 1 \not\sim 3$. Hence, without loss of generality, either (1,2,1) $\Delta_H(1') = \{2', 4'\}$ or (1,2,2) $\Delta_H(1') = \{4', 5'\}$.

In subcase (1,2,1), we have $\mu P\mathbf{e}_1 = P\mathbf{e}_{1'} + P\mathbf{e}_{2'} + P\mathbf{e}_{3'}$, whence

$$\mu^2 P\mathbf{e}_1 = P\mathbf{e}_1 + P\mathbf{e}_{2'} + P\mathbf{e}_{4'} + \mu P\mathbf{e}_{2'} + \mu P\mathbf{e}_{3'}$$

that is,

$$\mu^2 P\mathbf{e}_1 = P\mathbf{e}_1 + (\mu + 1)(\mu P\mathbf{e}_2 - \mathbf{v}_2) + \mu(\mu P\mathbf{e}_3 - \mathbf{v}_3) + \mu P\mathbf{e}_4 - \mathbf{v}_4. \quad (2)$$

Now a parity check shows that $\mu = 1$. (If $\mu = \pm 2$ then Eq.(2) can be written in the form $\Sigma_{i\in X} a_i P\mathbf{e}_i = \mathbf{0}$ with $\Sigma_{i\in X} a_i \not\equiv 0 \bmod 2$.) Hence

$$2\mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = 2P\mathbf{e}_2 + P\mathbf{e}_3 + P\mathbf{e}_4,$$

and this too contradicts Lemma 2.4

In subcase (1,2,2), again $\mu P\mathbf{e}_1 = P\mathbf{e}_{1'} + P\mathbf{e}_{2'} + P\mathbf{e}_{3'}$, and now

$$\mu^2 P\mathbf{e}_1 = P\mathbf{e}_1 + P\mathbf{e}_{4'} + P\mathbf{e}_{5'} + \mu P\mathbf{e}_{2'} + \mu P\mathbf{e}_{3'},$$

that is,

$$\mu^2 P\mathbf{e}_1 = P\mathbf{e}_1 + \mu P\mathbf{e}_4 - \mathbf{v}_4 + \mu P\mathbf{e}_5 - \mathbf{v}_5 + \mu(\mu P\mathbf{e}_2 - \mathbf{v}_2) + \mu(\mu P\mathbf{e}_3 - \mathbf{v}_3).$$

A parity check shows that $\mu = 1$. Hence

$$\mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 + \mathbf{v}_5 = P\mathbf{e}_2 + P\mathbf{e}_3 + P\mathbf{e}_4 + P\mathbf{e}_5,$$

and this contradicts Lemma 2.4.

It remains to consider case (2), where without loss of generality we take $|\Delta_H(1)| = |\Delta_H(2)| = 2$ and $\Delta_H(i) = \{i'\}$ $(i = 3, \ldots, t)$.

Lemma 3.1 *In Case (2), neither vertex 1 nor vertex 2 is adjacent to two vertices in* $\{3', 4', \ldots, t'\}$.

Proof. It suffices to rule out the case that $\Delta_H(2) = \{3', 4'\}$. Here we have $\mu P\mathbf{e}_2 = \mathbf{v}_2 + P\mathbf{e}_{3'} + P\mathbf{e}_{4'} = \mathbf{v}_2 + \mu P\mathbf{e}_3 - \mathbf{v}_3 + \mu P\mathbf{e}_4 - \mathbf{v}_4$. A parity check shows that $\mu = 1$. Hence

$$P\mathbf{e}_2 + \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{v}_2 + P\mathbf{e}_3 + P\mathbf{e}_4.$$

and this contradicts Lemma 2.4. □

In view of Lemma 3.1, we may assume that $\Delta_H(2) = \{2', 3'\}$. We distinguish two subcases: (2,1) $1 \not\sim 1'$, (2,2) $1 \sim 1'$. In subcase (2,1), we have $1 \sim 2'$ by Lemma 3.1. Moreover, since vertices 1 and 2 have distinct H-neighbourhoods, we may assume that $\Delta_H(1) = \{2', 4'\}$. Now we have

$$\mu P\mathbf{e}_1 = \mathbf{v}_1 + P\mathbf{e}_{2'} + P\mathbf{e}_{4'} = \mathbf{v}_1 + \mu P\mathbf{e}_2 - P\mathbf{e}_{3'} - \mathbf{v}_2 + \mu P\mathbf{e}_4 - \mathbf{v}_4$$

$$= \mathbf{v}_1 + \mu P\mathbf{e}_2 - \mu P\mathbf{e}_3 + \mathbf{v}_3 - \mathbf{v}_2 + \mu P\mathbf{e}_4 - \mathbf{v}_4.$$

If $\mu = 2$ then

$$2P\mathbf{e}_1 + 2P\mathbf{e}_3 + \mathbf{v}_2 + \mathbf{v}_4 = 2P\mathbf{e}_2 + 2P\mathbf{e}_4 + \mathbf{v}_1 + \mathbf{v}_3,$$

and we obtain a contradiction by equating coefficients of $P\mathbf{e}_1$.

If $\mu = -2$ then

$$2P\mathbf{e}_1 + 2P\mathbf{e}_3 + \mathbf{v}_1 + \mathbf{v}_3 = 2P\mathbf{e}_2 + 2P\mathbf{e}_4 + \mathbf{v}_2 + \mathbf{v}_4,$$

whence $\mathbf{v}_2 = P\mathbf{e}_1 + P\mathbf{e}_3$, a contradiction.

Hence $\mu = 1$ and we have

$$P\mathbf{e}_1 + P\mathbf{e}_3 + \mathbf{v}_2 + \mathbf{v}_4 = P\mathbf{e}_2 + P\mathbf{e}_4 + \mathbf{v}_1 + \mathbf{v}_3.$$

It follows that $\Delta_X(1) = \{3\}$, $\Delta_X(2) = \{4\}$, $\Delta_X(3) = \{1, h\}$ and $\Delta_X(4) = \{2, h\}$ for some $h > 4$. Without loss of generality, $h = 5$. Thus the vertices $1, 2, 3, 4, 5$ induce a path which is component of G_X, while any other component of G_X is a cycle.

By Theorem 2.1(ii), G has a 1-eigenvector $\mathbf{x} = (x(i))_{i \in V(G)}$ such that $x(1) = 1$ and $x(i) = 0$ $(i = 2, \dots, t)$. By Lemma 2.3, we have $x(i') = 0$ for all $i \geq 5$. Let $x(2') = a$, so that $x(3') = -a$ and $x(4') = 1 - a$. For $i = 2, 3, 4$, let $\Delta_H(i') = \{i''\}$. Then $x(2'') = a - 1$, $x(3'') = 0$ and $x(4'') = -a$. Since vertices $2', 3', 4'$ are endvertices of H, they constitute an independent set. Thus if $3' \sim 1'$ then $x(1') = 0$ and so $x(2'') = x(4'') = 0$, a contradiction. Hence $3' \sim j'$ for some $j \geq 5$ and we have:

$$P\mathbf{e}_2 = P\mathbf{e}_{2'} + P\mathbf{e}_{3'} + P\mathbf{e}_4 = P\mathbf{e}_1 - P\mathbf{e}_{4'} - P\mathbf{e}_3 + P\mathbf{e}_2 + P\mathbf{e}_3 + P\mathbf{e}_{j'} + P\mathbf{e}_4$$

$$= P\mathbf{e}_1 - P\mathbf{e}_4 + \mathbf{v}_4 - P\mathbf{e}_3 + P\mathbf{e}_2 + Pe_3 + P\mathbf{e}_j - \mathbf{v}_j + P\mathbf{e}_4.$$

Hence $\mathbf{v}_j = P\mathbf{e}_1 + P\mathbf{e}_j + \mathbf{v}_4$, a contradiction.

Now we turn to subcase (2,2), where $1' \sim 1 \not\sim 3'$ and we may assume that either (2,2,1) $1 \sim 2'$ or (2,2,2) $1 \sim 4'$. In subcase (2,2,1), H has degree sequence $1^{(2)}, 2^{(t-2)}$, and so H is a path; its endvertices are $2'$ and $3'$. Since $\Delta_H(2) = \{2', 3'\}$, the subgraph of G induced by $V(H) \dot\cup \{2\}$ is a $(t+1)$-cycle. By Lemma 2.5, this subgraph has μ as a simple eigenvalue, and so $\mu = \pm 2$.

Since $1'$ is not adjacent to both $2'$ and $3'$, we should consider just three possibilities: (2,2,1,1) $\Delta_H(1') = \{4', 5'\}$, (2,2,1,2) $\Delta_H(1') = \{2', 4'\}$, (2,2,1,3) $\Delta_H(1') = \{3', 4'\}$.

In subcase (2,2,1,1) we have $\mu P\mathbf{e}_1 = \mathbf{v}_1 + P\mathbf{e}_{1'} + P\mathbf{e}_{2'}$, whence

$$\mu^2 P\mathbf{e}_1 = \mu\mathbf{v}_1 + P\mathbf{e}_1 + P\mathbf{e}_{4'} + P\mathbf{e}_{5'} + \mu P\mathbf{e}_{2'}$$

$$= \mu\mathbf{v}_1 + P\mathbf{e}_1 + \mu P\mathbf{e}_4 - \mathbf{v}_4 + \mu P\mathbf{e}_5 - \mathbf{v}_5 + \mu(\mu P\mathbf{e}_2 - \mathbf{v}_2 - P\mathbf{e}_{3'})$$

$$= \mu\mathbf{v}_1 + P\mathbf{e}_1 + \mu P\mathbf{e}_4 - \mathbf{v}_4 + \mu P\mathbf{e}_5 - \mathbf{v}_5 + \mu^2 P\mathbf{e}_2 - \mu\mathbf{v}_2 - \mu(\mu P\mathbf{e}_3 - \mathbf{v}_3).$$

Now a parity check gives a contradiction.

In subcase (2,2,1,2), we have $\mu P\mathbf{e}_1 = \mathbf{v}_1 + P\mathbf{e}_{1'} + P\mathbf{e}_{2'}$, and so

$$\mu^2 P\mathbf{e}_1 = \mu\mathbf{v}_1 + P\mathbf{e}_1 + P\mathbf{e}_{2'} + P\mathbf{e}_{4'} + \mu P\mathbf{e}_{2'} = \mu\mathbf{v}_1 + P\mathbf{e}_1 + \mu P\mathbf{e}_4 - \mathbf{v}_4 + (\mu+1)P\mathbf{e}_{2'}$$

$$= \mu\mathbf{v}_1 + P\mathbf{e}_1 + \mu P\mathbf{e}_4 - \mathbf{v}_4 + (\mu+1)(\mu P\mathbf{e}_2 - \mathbf{v}_2 - P\mathbf{e}_{3'})$$

$$= \mu\mathbf{v}_1 + P\mathbf{e}_1 + \mu P\mathbf{e}_4 - \mathbf{v}_4 + (\mu+1)(\mu P\mathbf{e}_2 - \mathbf{v}_2 - \mu P\mathbf{e}_3 + \mathbf{v}_3).$$

If $\mu = 2$ then

$$3P\mathbf{e}_1 + \mathbf{v}_4 + 3\mathbf{v}_2 + 6P\mathbf{e}_3 = 2\mathbf{v}_1 + 2P\mathbf{e}_4 + 6P\mathbf{e}_2 + 3\mathbf{v}_3.$$

If $\mu = -2$ then

$$3P\mathbf{e}_1 + 2\mathbf{v}_1 + 2P\mathbf{e}_4 + \mathbf{v}_4 + 2P\mathbf{e}_3 + \mathbf{v}_3 = 2P\mathbf{e}_2 + \mathbf{v}_2.$$

For both values of μ, Lemma 2.4 is contradicted.

In subcase (2,2,1,3), we have $\mu P\mathbf{e}_1 = \mathbf{v}_1 + P\mathbf{e}_{1'} + P\mathbf{e}_{2'}$ and so

$$\mu^2 P\mathbf{e}_1 = \mu\mathbf{v}_1 + P\mathbf{e}_1 + P\mathbf{e}_{3'} + P\mathbf{e}_{4'} + \mu P\mathbf{e}_{2'}$$

$$= \mu\mathbf{v}_1 + P\mathbf{e}_1 + \mu P\mathbf{e}_3 - \mathbf{v}_3 + \mu P\mathbf{e}_4 - \mathbf{v}_4 + \mu P\mathbf{e}_{2'}$$

$$= \mu\mathbf{v}_1 + P\mathbf{e}_1 + \mu P\mathbf{e}_3 - \mathbf{v}_3 + \mu P\mathbf{e}_4 - \mathbf{v}_4 + \mu(\mu P\mathbf{e}_2 - \mathbf{v}_2 - \mu P\mathbf{e}_3 + \mathbf{v}_3).$$

Again a parity check gives a contradiction.

Now we consider subcase (2,2,2), where $1 \sim 4'$ and H is a path with end-vertices $3'$ and $4'$. By Lemma 2.5 the subgraph of G induced by $V(H) \dot\cup \{3,4\}$ has μ as a double eigenvalue; hence this subgraph is a $(t{+}2)$-cycle, and $\mu = 1$. Let $\Delta_H(3') = \{i'\}$, and let H_i be the subgraph induced by $V(H) \dot\cup \{i\}$. Then $i \in \{1,2\}$ for otherwise H_i is a tree without a 1-eigenvector $\mathbf{x}$ such that $x(i) = 1$. Similarly, $\Delta_H(4') = \{j'\}$, where $j \in \{1,2\}$. Since $t > 3$ we have $i \neq j$, and so either (2,2,2,1) $\Delta_X(3') = \{2'\}, \Delta_X(4') = \{1'\}$ or (2,2,2,2) $\Delta_X(3') = \{1'\}, \Delta_X(4') = \{2'\}$.

In subcase (2,2,2,1), we have $\mu P\mathbf{e}_4 = P\mathbf{e}_{4'} + \mathbf{v}_4$, whence

$$\mu^2 P\mathbf{e}_4 = P\mathbf{e}_4 + P\mathbf{e}_1 + P\mathbf{e}_{1'} + \mu\mathbf{v}_4 = P\mathbf{e}_4 + P\mathbf{e}_1 + \mu P\mathbf{e}_1 - P\mathbf{e}_{4'} - \mathbf{v}_1 + \mu\mathbf{v}_4$$

$$= P\mathbf{e}_4 + P\mathbf{e}_1 + \mu P\mathbf{e}_1 - \mu P\mathbf{e}_4 + \mathbf{v}_4 - \mathbf{v}_1 + \mu\mathbf{v}_4.$$

Since $\mu = 1$, we have

$$P\mathbf{e}_4 + \mathbf{v}_1 = 2P\mathbf{e}_1 + 2\mathbf{v}_4,$$

contradicting Lemma 2.4.

In subcase (2,2,2,2), we have $\mu P\mathbf{e}_4 = P\mathbf{e}_{4'} + \mathbf{v}_4$ and

$$\mu^2 P\mathbf{e}_4 = P\mathbf{e}_4 + P\mathbf{e}_1 + P\mathbf{e}_{2'} + \mu\mathbf{v}_4 = P\mathbf{e}_4 + P\mathbf{e}_1 + \mu P\mathbf{e}_2 - P\mathbf{e}_{3'} - \mathbf{v}_2 + \mu\mathbf{v}_4$$

$$= P\mathbf{e}_4 + P\mathbf{e}_1 + \mu P\mathbf{e}_2 - \mu P\mathbf{e}_3 + \mathbf{v}_3 - \mathbf{v}_2 + \mu\mathbf{v}_4.$$

Since $\mu = 1$, we have

$$P\mathbf{e}_3 + \mathbf{v}_2 = P\mathbf{e}_1 + P\mathbf{e}_2 + \mathbf{v}_3 + \mathbf{v}_4,$$

contradicting Lemma 2.4.

We have now proved:

Proposition 3.2. *Let G be a connected cubic graph of order n with an eigenvalue μ of multiplicity $\frac{1}{2}n$. If $\mu \notin \{-1, 0\}$ then $\mu = 1$, $n = 10$ and G is the Petersen graph.*

4 The case $\mu \in \{-1, 0\}$

In this section we assume that G is a connected cubic graph, with $\mu \in \{-1, 0\}$ and $k = \frac{1}{2}n + 1$ (that is, $k = t + 2$). By Lemma 2.6, we know that G has a connected star complement for μ, say $H = G - X$. By Proposition 2.7, H is a tree; moreover $|\Delta_H(u)| = 1$ for all $u \in X$, and so G_X is a union of disjoint cycles. Note that there exist (at least) two vertices in X with a common neighbour in H.

Lemma 4.1. *Let G be graph with X as a star set for the eigenvalue μ, and let $H = G - X$. Suppose that u, v are distinct vertices in X such that $\Delta_H(u) = \Delta_H(v)$.*
(i) *If $\mu = -1$ then $\Delta_X(u) \dot\cup \{u\} = \Delta_X(v) \dot\cup \{v\}$ (and so u, v are co-duplicate vertices).*
(ii) *If $\mu = 0$ then $\Delta_X(u) = \Delta_X(v)$ (and so u, v are duplicate vertices).*

Proof. Both (i) and (ii) follow from Lemma 2.4 and the relation

$$\mu P\mathbf{e}_u - \Sigma_{i\in\Delta_X(u)} P\mathbf{e}_i = \mu P\mathbf{e}_v - \Sigma_{j\in\Delta_X(v)} P\mathbf{e}_j.$$

$\square$

Let $X = \{1, 2, \ldots, t+2\}$, $\overline{X} = \{1', 2', \ldots, t'\}$, with $\Delta_H(1) = \Delta_H(2) = \{1'\}$. Suppose first that $\mu = -1$. By Lemma 4.1(i), we have $1 \sim 2$, and we may take $\Delta_X(1) = \{2, 3\}$, $\Delta_X(2) = \{1, 3\}$. This argument shows that no vertex of H is adjacent to two vertices in different components of G_X

If $3 \sim 1'$ then $G = K_4$, and so we suppose that $3 \sim 2'$. By Theorem 2.1(ii), G has a (-1)-eigenvector $\mathbf{x}$ with $x(1) = 1$ and $x(i) = 0$ ($i = 2, 3, \ldots, t+2$). We have $x(1') = x(2') = -1$. Consider an r-cycle C other than 1231 in G_X. If C has two vertices with a common neighbour in H then $r = 3$, and by Lemma 2.3, $x(i') = 0$ for each neighbour i' in H of a vertex of C. The same conclusion holds when C does not have two vertices with a common neighbour in H. It follows that $x(i') = 0$ ($i = 3, \ldots, t$). Thus the non-zero entries of $\mathbf{x}$ are $1, -1, -1$, and $\mathbf{x}$ is not orthogonal to the all-1 vector $\mathbf{j} \in I\!R^n$. This is a contradiction because $\mathbf{j}$ is a 3-eigenvector of G.

Next suppose that $\mu = 0$. By Lemma 4.1(ii), we may take $\Delta_X(1) = \Delta_X(2) = \{3, 4\}$, where $3 \not\sim 1' \not\sim 4$; moreover, $3 \not\sim 4$ because $\Delta_H(4) \neq \emptyset$. Note that again no vertex of H is adjacent to two vertices in different components of G_X. Now let $\mathbf{x}$ be a 0-eigenvector with $x(1) = 1$ and $x(i) = 0$ ($i = 2, \ldots, t+2$). Note that $x(1') = 0$, and consider an r-cycle C other than 13241 in G_X. If C has two vertices with a common neighbour in H then $r = 4$, and by Lemma 2.3, $x(i') = 0$ for each neighbour i' in H of a vertex in C. The same conclusion holds when C does not have two vertices with a common neighbour in H.

If vertices 3 and 4 have a common neighbour in H, say $2'$, then $x(2') = -1$; moreover if $\Delta_H(1') = \{j'\}$ then $x(j') = -1$, while $x(i') = 0$ ($i = 3, \ldots, t$). In this case, $j = 2$ and $G = \overline{2K_3}$. If vertices 3 and 4 have different neighbours in H, say $\Delta_H(3) = \{2'\}$ and $\Delta_H(4) = \{3'\}$ then $x(2') = x(3') = -1$, while $x(i') = 0$ ($i = 4, \ldots, t$). Now $\mathbf{j}^\perp\mathbf{x} \neq 0$, a contradiction as before. We have therefore proved:

Proposition 4.2. *Let G be a connected cubic graph of order n with an eigenvalue μ of multiplicity $\frac{1}{2}n + 1$. If $\mu = -1$ then $G = K_4$, and if $\mu = 0$ then $G = \overline{2K_3}$.*

In view of Lemma 2.6, we can combine Propositions 2.7, 3.2 and 4.2 to obtain Theorem 1.1.

References

[1] F. K. Bell and P. Rowlinson, On the multiplicities of graph eigenvalues, *Bull. London Math. Soc.* 35 (2003), 401-408.

[2] N. E. Clarke, W. D. Garraway, C. A. Hickman, R. J. Nowakowski, Graphs where star sets are matched to their complements, *J. Combin. Math. Combin. Comput.* 37 (2001) 177-185.

[3] D. Cvetković, P. Rowlinson and S. K. Simić, *An Introduction to the Theory of Graph Spectra*, Cambridge University Press (Cambridge), 2010.

[4] P. Rowlinson, Eutactic stars and graph spectra, in: R. A. Brualdi, S. Friedland, V. Klee (Eds.), Combinatorial and Graph-Thoeretical Problems in Linear Algebra, Springer-Verlag, New York, 1993, pp.153-164.