SOME CLASSES OF TOPOLOGICAL VECTOR SPACES

ASSOCIATED WITH THE CLOSED GRAPH THEOREM

by

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CHAPTER I

PRELIMINARIES

1.1 Introduction

Given two topological vector spaces E and F and a linear mapping $t : E \to F$ with a closed graph, t may or may not be continuous. When such a linear mapping t is necessarily continuous, the closed graph theorem is said to hold for E and F. For example, if E and F are Banach spaces, then every linear mapping with a closed graph of E into F is necessarily continuous.

The main aim of this thesis is to give precise descriptions of certain topological vector spaces that can serve as domain spaces, and also those that can serve as range spaces for a closed graph theorem. This is motivated by the works of M. Mahowald [35], N. Adasch [1], V. Eberhardt [12, 14, 11] and N.J. Kalton [25].

Chapter 2 of the thesis is concerned with the concept of essential separability which turns out to be a useful variation of separability. We look at various characterizations of essential separability and link it up with the well-known concepts of weak compactness and weak relative compactness (Section 2.2).

In Chapter 3, we introduce the class of δ-barrelled spaces which serve as domain spaces for some closed graph theorems (Theorems 3.1.2 and 4.1.3). We show that in the separated case, δ-barrelled spaces can be characterized in terms of essential separability (Theorem 3.1.1). We establish also some of the basic permanence properties of δ-barrelled spaces including the countable codimensional subspace property (Theorem 3.1.6). It is seen that the class of separated δ-barrelled spaces is a proper subclass of Kalton's domain spaces and strictly contains the class of separated barrelled spaces (Example 3.1.1(a), (d)). Also in this Chapter, conditions under which a δ-barrelled space is barrelled are considered.

In Chapter 4, those locally convex spaces which can serve as range spaces in our closed graph theorem in which the domain space is an arbitrary δ-barrelled space with its Mackey topology are considered. These are the infra-δ-spaces. We also look at the domain spaces (δ-spaces) for the corresponding open mapping theorem (Theorem 4.1.4).

Finally, Chapter 5 deals with some topics that are closely related to the concept of δ-barrelledness. In particular, we look at the closed graph theorem when the range space is not assumed to be complete. Then we generalize δ-barrelledness to δ-ultrabarrelledness in general topological vector spaces. In a way similar to the characterization of δ-barrelledness, we obtain a characterization of δ-ultrabarrelledness by means of a closed graph theorem (Theorem 5.2.2). We end the Chapter with a generalization of some of our concepts to arbitrary infinite cardinals.

1.2 Notations

Let E be a vector space over the field K of real or complex numbers with its usual topology. The vector space E together with a vector space topology ξ is called a topological vector space and denoted (E, ξ) or simply by E if it is unnecessary to name the topology. We shall denote the algebraic dual of E by E^* and if (E, ξ) is a separated locally convex space, its (continuous) dual will be denoted by E' .

If A is a subset of a locally convex space E, the (absolute) polar of A in E^* (respectively E') will be denoted by $A^\bullet$ (respectively A^o). The (linear) dimension of E will often be denoted by $\dim E$. The topology induced on a subset A of a topological space $(X, \mathcal{J})$ will be denoted by $\mathcal{J}|_A$ and the cardinality of a set S will be written as $|S|$.

Let E be a separated locally convex space with dual E'. Then $\sigma(E, E')$, $\tau(E, E')$ and $\beta(E, E')$ denote respectively the weak, Mackey and strong topologies on E determined by E'..

Generally, we follow the topological vector space notation of [43] except that by a Mackey space E we mean a locally convex space endowed with its Mackey topology $\tau(E, E')$. Also, throughout, $\mathbb{R}$ and $\mathbb{C}$ will denote the fields of real and complex numbers respectively and $\mathbb{N}$ will denote the natural numbers. K will be used to denote the field of real or complex numbers when it is not necessary to specify the particular one. $C(X)$ will denote the space of scalar-valued continuous functions on a completely regular space X and $C_c(X)$ will denote $C(X)$ with the topology of compact convergence. The dual space of $C_c(X)$ will be denoted by $C(X)'$ and $[C(X)']_\sigma$ will represent $C(X)'$ under its weak topology $\sigma(C(X)', C(X))$. We write c and $\aleph_0$ for $|\mathbb{R}|$ and $|\mathbb{N}|$ respectively.

1.3 Some Classes of Topological Vector Spaces

In a locally convex space E, a closed absolutely convex absorbent subset B is called a barrel and E is said to be barrelled if every barrel is a neighbourhood of the origin in E. More often than not, the dual characterization of barrelledness is used in practice. This states that a separated locally convex space E is barrelled if and only if every $\sigma(E', E)$- bounded set is equicontinuous (see for example [43, Chapter IV §1]).

Locally convex spaces of the second category are barrelled. In particular, Fréchet spaces and so Banach spaces are barrelled. The inductive limit of barrelled spaces is again barrelled and a separated product of barrelled spaces is also barrelled. Also the completion of a barrelled space is a barrelled space. A closed vector subspace of a barrelled space need not be barrelled. However any subspace of countable codimension of a barrelled space is of the same type [46, 54].

According to T. Husain [20], a separated locally convex space (E, ξ) is countably barrelled if each $\sigma(E', E)$ bounded set which is the union of a sequence of ξ-equicontinuous sets is ξ-equicontinuous. Later M. de Wilde and C. Houet in [9] defined σ-barrelled spaces as those separated locally convex spaces (E, η) for which whenever (x'_n) is a $\sigma(E', E)$-bounded sequence, the set $\{x'_n : n \in \mathbb{N}\}$ is η-equicontinuous. About the same time as De Wilde and Houet, the σ-barrelled spaces were also considered independently by M. Levin and S. Saxon [31] who called them ω-barrelled spaces.

Every countably barrelled space is σ-barrelled. Countably barrelled spaces and σ-barrelled spaces have some of the basic permanence properties that barrelled spaces have including the countable codimensional subspace property [56, Theorem 6; 31, §4 Theorem]. A separable σ-barrelled space is barrelled [9, Corollary 4a]. In particular, every separable countably barrelled space is barrelled.

The concept of barrelled spaces is generalized to ultrabarrelled spaces in the general topological vector space setting. As defined by W. Robertson in [44], ultrabarrelled spaces are those topological vector spaces (E, ξ) for which any vector space topology on E with a base of ξ-closed neighbourhoods of the origin is necessarily coarser than ξ. A closed balanced subset B of a topological vector space E is called an ultrabarrel if there exists a sequence (B_n) of balanced absorbent subsets of E such that $B_1 + B_1 \subseteq B$ and $B_{n+1} + B_{n+1} \subseteq B_n$ for each $n \in \mathbb{N}$. Such a sequence (B_n) is called a defining sequence for B. A topological vector space (E, ξ) is ultrabarrelled if and only if every ultrabarrel in E is an ξ-neighbourhood of the origin in E [21]. Every locally convex ultrabarrelled space is barrelled but the converse is not necessarily true (44, page 256).

The notion of fully complete spaces was introduced by Collins in [8]. V. Pták called such spaces B-complete and introduced the related class of B_r-complete spaces in [42]. A separated locally convex space E with dual E' is said to be B-complete if every vector subspace A of E' such that $A \cap U^{\circ}$ is $\sigma(E', E)$-closed for each neighbourhood U of the origin is necessarily $\sigma(E', E)$-closed; E is B_r-complete if this holds for $\sigma(E', E)$-dense subspaces (in which case the subspace coincides with E').

Every B-complete space is B_r-complete and every B_r-complete space is complete. A complete space need not be B-complete (see for example [43, Chapter VI, supplement (1)]). Every Fréchet space is B-complete and so in particular every Banach space is B-complete.

Closely related to the notions of B-completeness and B_r-completeness are the concepts of t-polar and weakly t-polar spaces introduced by Persson in [38]. A separated locally convex space E is said to be t-polar if each vector subspace H of E' is $\sigma(E', E)$-closed whenever $H \cap B^{\circ}$ is $\sigma(E', E)$-closed for every barrel B in E. It is weakly t-polar if this property holds for $\sigma(E', E)$-dense vector subspaces H of E'. Every B-complete (respectively B_r-complete) space is t-polar (respectively weakly t-polar). As shown in [38], there exist t-polar Mackey spaces which are not B-complete and not even complete. However for barrelled spaces, these concepts coincide.

The following definitions of s-spaces and infra-s-spaces are due to N. Adasch [1]. Let E be a separated locally convex space with dual E'. Let H be a vector subspace of E'. Denote by $\widehat{H}$ the intersection of all the $\sigma(E^*, E)$- quasi-closed subspaces of E^* which contain H. E is called an s-space if for each vector subspace H of E' we have $\widehat{H} \cap E' = \overline{H}$, the $\sigma(E', E)$-closure of H. It is called an infra-s-space if for each $\sigma(E', E)$-dense vector subspace H of E', we have $\widehat{H} \cap E' = E'$. As pointed out in [1, §5], every (weakly) t-polar space is an (infra-)s-space. Examples are given in [1, §6] to show that there are s-spaces which are not t-polar and also that an infra-s-space need not be weakly t-polar.

These various classes of topological vector spaces we have been discussing in this section are linked up by closed graph theorems and open mapping theorems which have proved to be useful tools in the study of Functional Analysis. In [42], V. Pták showed that a linear mapping with a closed graph of a barrelled space into a B_r-complete space F is continuous and that a linear mapping with a closed graph of a B-complete space onto a barrelled space is open. Thus in particular the closed graph theorem holds for the case when E is barrelled and F is a Banach space. Later Mahowald [35] established the converse that a separated locally convex space E is barrelled if whenever $t : E \to F$ is a linear mapping with a closed graph of E into an arbitrary Banach space F then t is continuous.

A. Persson has shown in [38, Theorem 1], that a linear mapping with a closed graph of a barrelled space into a weakly t-polar space is continuous. He proved also that a linear mapping with a closed graph of a t-polar space onto a barrelled space is open [38, Theorem 2'].

Following Mahowald's characterization of barrelled spaces by means of the closed graph theorem, there has been growing interest in the question of characterizations of other topological vector spaces. N. Adasch showed in [1, §3,(2)] that a separated locally convex space F is an infra-s-space if and only if whenever $t: E \to F$ is a linear mapping with a closed graph of a barrelled space E into F, then t is continuous.

In the non-locally convex case, Iyahen [21, Theorem 3.2], gave a characterization of the ultrabarrelled spaces. He showed that a topological vector space E is ultrabarrelled if and only if every linear mapping with a closed graph of E into any complete metric linear space is continuous.

N.J. Kalton in [25], defined the class $\mathcal{C}(\zeta_B)$ (respectively $\mathcal{C}(\zeta)$) of locally convex spaces as those which can serve as domain spaces for a closed graph theorem in which the range space is an arbitrary separable Banach (respectively separable B_r-complete) space. He also gave an example of a Mackey space in $\mathcal{C}(\zeta)$ which is not barrelled.

V. Eberhardt [14, 11] has also described those spaces (GN-spaces) that serve as domain spaces for a closed graph theorem in which the range space is a normed space. The corresponding domain spaces (GM-spaces) for metrizable range spaces are discussed by Eberhardt and Roelcke in [15] (see also [11]).

1.4 Dense Subsets of Products

We recall that a subset of a topological space is said to be separable if it contains an at most countable dense subset. Every open subset of a separable set is again separable. The fact that a subset of a non-metrizable separable set need not be separable [10, Chapter VIII, Theorem 7.2 (2)], limits the use of separability. In Chapter 2 we introduce a variant of the idea of separability which has the hereditary property and which we can apply in Chapter 3 to the study of the closed graph theorem.

Let K be the field of real or complex numbers and let M be a non-empty set. We denote by K^M (respectively $K^{(M)}$) the product $\prod_{\mu\in M} K$ (respectively direct sum $\bigoplus_{\mu\in M} K$) of copies of K indexed by M. The space K^M (respectively $K^{(M)}$) with its product (respectively direct sum) topology is a topological vector space. We state the following result which is proved in [10, Chapter VIII, Theorem 7.2] for reference purposes.

Theorem 1.4.1

Let $X_\mu (\mu\in M)$ be non-empty Hausdorff topological spaces where M is an index set. Then $\prod_{\mu\in M} X_\mu$ is separable if and only if each X_μ is separable and all but at most c are spaces consisting of a single point.

As a special case of Theorem 1.4.1, we have the following:

Corollary

Let M be a non-empty set and let K be the field of real or complex numbers with its usual topology. Then K^M is separable if and only if $|M| \le c$.

We note that by [10, Chapter VIII, §7, Ex.4], the cardinality of any subset of a separable space is at most 2^c. We shall require the following partial extension of the Corollary to Theorem 1.4.1 above. The proof uses the techniques of the proof of part (3) of [10, Chapter VIII, Theorem 7.2].

Theorem 1.4.2

Let M be a set such that $|M| > 2^c$. Then K^M cannot have a dense subset of cardinality at most c.

Proof

Suppose on the contrary that K^M has a dense subset A of cardinality at most c. For each $\mu \in M$, choose two disjoint non-empty open subsets U_μ, V_μ of K. Let $A_\mu = A \cap p_\mu^{-1}(U_\mu)$, where p_μ is the projection of K^M onto the μth component space.

Consider the mapping ϕ of M into $\mathcal{P}(A)$, the power set of A, defined by $\phi(\mu) = A_\mu$. We show that ϕ is one-to-one. If $\mu, \nu \in M$ and $\mu \neq \nu$, let U_μ, V_μ, U_ν and V_ν be as above. Since $p_\mu^{-1}(U_\mu) \cap p_\nu^{-1}(V_\nu)$ is a non-empty open set in K^M and since A is dense in K^M, there exists $a \in K^M$ such that $a \in A \cap p_\mu^{-1}(U_\mu) \cap p_\nu^{-1}(V_\nu)$. Thus $a \in A \cap p_\mu^{-1}(U_\mu) = A_\mu$ and $a \notin A_\nu = A \cap p_\nu^{-1}(U_\nu)$, since $U_\nu \cap V_\nu = \emptyset$. Hence $A_\nu \neq A_\mu$.

Since $\phi : M \to \mathcal{P}(A)$ is one-to-one, it follows that $|M| \leq |\mathcal{P}(A)| \leq 2^c$ ([10, Chapter II, 7.2]). This contradiction establishes the result.

1.5 Linear Dimension

We recollect the following definitions. A non-empty subset X of a vector space E is said to be linearly independent if for every finite sequence $x_1, \ldots, x_n$ of distinct elements of X, whenever

$\sum_{r=1}^{n} \alpha_r x_r = 0$ we have that the scalars $\alpha_r = 0$, $(r = 1, \ldots, n)$. A maximal linearly independent subset of a non-zero vector space E is called a basis of E; every such vector space E has a basis and all bases of E have the same cardinality which is called the (linear) dimension of E. By convention the dimension of the zero vector space is zero. Whenever we talk of the dimension of a (topological) vector space we shall mean the linear dimension of the vector space.

We shall need the following well-known result in our subsequent discussions. We give a proof for completeness.

<u>Lemma 1.5.1</u>

Let A and B be vector subspaces of a vector space E such that $A \subseteq B$. Then $\dim E/_B \leq \dim E/_A$.

<u>Proof</u>

Consider the mapping $s : E/_A \to E/_B$ defined by $s(x + A) = x + B$. Since $A \subseteq B$, we have that s is well-defined. It is clear that s is also linear and onto. Suppose that $x_1 + B, \ldots, x_n + B$ are linearly independent in $E/_B$. Then if $\sum_{r=1}^{n} \alpha_r (x_r + A) = 0 \in E/_A$, we have on applying s to both sides that $\sum_{r=1}^{n} \alpha_r (x_r + B) = 0 \in E/_B$ and so $\alpha_r = 0$ $(r = 1, \ldots, n)$. Since $x_1 + A, \ldots, x_n + A$ are then linearly independent in E/A, it follows that $\dim E/_A \geq \dim E/_B$.

We note that if X is a topological vector space of dimension α over K, then X^* under $\sigma(X^*, X)$ is topologically isomorphic to K^M, where $|M| = \alpha$ (see for example [28, §9, 1(3)] and [43, Chapter II, Proposition 12]). We use this observation together with the Corollary of Theorem 1.4.1 to obtain the following result.

Theorem 1.5.1

Let E be a topological vector space over K. Then E^* is separable under $\sigma(E^*, E)$ if and only if the dimension of E is at most c.

Proof

By the above observation, $(E^*, \sigma(E^*, E))$ is topologically isomorphic to K^M under its product topology, where $|M| = \dim E$. Thus E^* is separable under $\sigma(E^*, E)$ if and only if K^M is separable, which by the Corollary of Theorem 1.4.1 is equivalent to $|M| \leq c$.

In [34, Theorem I - 1] G.W. Mackey showed that the dimension of an infinite dimensional Banach space F is at least c and if F is in addition separable then the dimension is precisely c. Recently, H.E. Lacey [30] has also shown by an elegant method that any infinite dimensional separable Banach space has dimension c.

We know also that the dimension of any infinite dimensional Fréchet space is at least c (see for example [5, Chapter II, §5, Exercise 24]). These results on the dimensions of infinite dimensional Banach spaces and infinite dimensional Fréchet spaces are established by duality arguments. It is perhaps surprising that the corresponding result for complete metrizable topological vector spaces can be deduced from these known results. We have the following theorem.

Theorem 1.5.2

The dimension of an infinite dimensional complete metrizable topological vector space is at least c. If in addition the space is separable, its dimension is precisely c.

Proof

Let (E, ξ) be an infinite dimensional complete metrizable topological vector space and let (x_n) be a sequence of linearly independent elements of E. Let p be an F-norm [28, §15, 11 (2)] on E defining ξ. For each n, choose $\alpha_n > 0$ such that $p(\alpha_n x_n) \leq \frac{1}{2^n}$ and put $y_n = \alpha_n x_n$. Let B be the absolutely convex envelope of $\{y_n : n \in \mathbb{N}\}$.

We show that B is ξ-bounded. Suppose that (λ_n) is a sequence of scalars with only finitely many non-zero terms and such that $\sum_{n=1}^{\infty} |\lambda_n| \leq 1$. For each $m \in \mathbb{N}$ we have

$$p\left(\sum_{n=m}^{\infty} \lambda_n y_n\right) \leq \sum_{n=m}^{\infty} p(\lambda_n y_n) \leq \sum_{n=m}^{\infty} p(y_n) \leq \frac{1}{2^{m-1}} .$$

Given $\epsilon>0$, we may therefore choose $M>1$ such that $p\left(\sum_{n=M}^{\infty} \lambda_n y_n\right) \leq \frac{\epsilon}{2}$ for all such (λ_n). We deduce from [26, 7.3] that the set $A = \{ \sum_{n=1}^{M-1} \mu_n y_n : \sum_{n=1}^{M-1} |\mu_n| \leq 1\}$ is ξ-bounded. Consequently there exists $\beta \geq 1$ such that $A \subseteq \beta\{x \in E : p(x) \leq \frac{\epsilon}{2}\}$. Thus for such a sequence (λ_n) of scalars, we have

$$p\left(\frac{1}{\beta} \sum_{n=1}^{\infty} \lambda_n y_n\right) \leq p\left(\frac{1}{\beta} \sum_{n=1}^{M-1} \lambda_n y_n\right) + p\left(\frac{1}{\beta} \sum_{n=M}^{\infty} \lambda_n y_n\right) \leq \frac{\epsilon}{2} + p\left(\sum_{n=M}^{\infty} \lambda_n y_n\right) \leq \epsilon ,$$

and consequently $B \subseteq \beta\{x \in E : p(x) \leq \epsilon\}$.

By [26, 5.2, 6.2], we have that the ξ-closure D of B is also ξ-bounded and absolutely convex. Let H be the linear span of D. The gauge of D is a norm on H and since D is absorbed by each ξ-neighbourhood of the origin, the resulting norm topology η is finer than the topology induced on H by ξ. We now show that H is complete under η. Let (z_n) be an η-Cauchy sequence. It is also an ξ-Cauchy sequence and so converges under ξ to $z_0 \in E$ say. Since $\{z_n : n \in \mathbb{N}\}$ is absorbed by D and since D is ξ-closed, it follows that $z_0 \in H$. Because η has a base of neighbourhoods of the origin which are ξ-closed sets, we may now deduce from [28, §18, 4(4)] that (z_n) converges under η to z_0.

Since each x_n is an element of H, the Banach space (H, η) is infinite dimensional. We therefore have that $\dim E \geq \dim H \geq c$. When E is separable its cardinality is c. This implies that the dimension of E cannot exceed c[33, Satz 2] and so must be c.

CHAPTER II

ESSENTIAL SEPARABILITY

Certain topological properties are inherited by subsets of sets in topological vector spaces. For example, subsets of bounded sets are also bounded. Unfortunately, separability is not one of such properties. The notion of essential separability defined below is intended to cater for this defect. This notion is introduced in this Chapter and various characterizations of it are also given (Section 2.1). We also link it up with the concepts of weak compactness and relative weak compactness which are well-known (Section 2.2).

2.1 Definition and General Properties

Let (E, F) be a dual pair. A subset A of E (regarded as a vector subspace of F^*) is said to be essentially separable for the dual pair (E,F) if it is contained in a $\sigma(F^*, F)$-separable set.

When the dual pair is clearly indicated, we simply say that A is essentially separable. For example, if E is a separated locally convex space with dual E' and A and B are subsets of E and E' respectively, "A (respectively B) is essentially separable for the dual pair (E, E') (respectively (E', E))" will usually be written as "A (respectively B) is essentially separable".

For a dual pair (E,F), any $\sigma(E, F)$-separable set or its subset is essentially separable. Although separability does not generally pass to subsets, each subset of an essentially separable set is again essentially separable.

If (E, F) is a dual pair such that F has dimension at most c, then any subset of E is essentially separable, for by Theorem 1.5.1, we must have that F^* is separable under $\sigma(F^*, F)$.

The following lemma gives us further examples of essentially separable sets in a given dual pair.

<u>Lemma 2.1.1</u>

Let (E, F) be a dual pair and let A be a non-empty essentially separable subset of E . If G is the $\sigma(E, F)$-closed vector subspace of E generated by A , then each subset of G is essentially separable for the dual pair (E, F) . If A_n $(n \in \mathbb{N})$ is an essentially separable subset of E , then so also is $\bigcup_{n=1}^{\infty} A_n$.

<u>Proof</u>

Let X be a $\sigma(F^*, F)$-separable set containing A and let $\{x_n^* : n \in \mathbb{N}\}$ be an at most countable $\sigma(F^*, F)$-dense subset of X . Then the $\sigma(F^*, F)$-closed vector subspace generated by $\{x_n^* : n \in \mathbb{N}\}$ is $\sigma(F^*, F)$-separable and contains G . Hence G and consequently any subset of it are essentially separable for the dual pair (E, F) .

To establish the second part, for each $n \in \mathbb{N}$ let B_n be a $\sigma(F^*, F)$-separable set such that $A_n \subseteq B_n$. Then since a countable union of separable sets is separable, $B = \bigcup_{n=1}^{\infty} B_n$ is $\sigma(F^*, F)$-separable and $\bigcup_{n=1}^{\infty} A_n \subseteq B$. Consequently, $\bigcup_{n=1}^{\infty} A_n$ is essentially separable.

Given a non-empty subset X of a vector space E , the linear span Y of X is a vector subspace of E . If (E, F) is a dual pair, then Y^{o} , the polar of Y in F , is a $\sigma(F, E)$-closed vector subspace of F and we can form the quotient space $F/_{Y^o}$ so that $(Y, F/_{Y^o})$ is a dual pair. We give next a useful characterization of essential separability in terms of the dimension of such a quotient space.

Theorem 2.1.1

Let (E, F) be a dual pair. Let A be a non-empty subset of E and let H be the linear span of A. Then A is essentially separable if and only if $\dim F/_{H^\circ} \leq c$.

Proof

We observe first that $(F/_{H^\circ})^*$ is topologically isomorphic to the $\sigma(F^*, F)$-closure G of H.

Suppose that $\dim F/_{H^\circ} \leq c$. Then by Theorem 1.5.1., we have that G is $\sigma(G, F/_{H^\circ})$-separable. Since $\sigma(F^*, F)$ and $\sigma(G, F/_{H^\circ})$ coincide on G, it follows that A is essentially separable.

If conversely we suppose that A is essentially separable, then there exists a $\sigma(F^*, F)$-separable set B which contains A. Let L be the $\sigma(F^*, F)$-closed vector subspace generated by B. Now $(F/_{L^\circ})^*$ is topologically isomorphic to L which is $\sigma(F^*, F)$-separable. Again from Theorem 1.5.1, it follows that $\dim F/_{L^\circ} \leq c$. Further, since $H \subseteq L$, we have that $L^\circ \subseteq H^\circ$ and consequently by Lemma 1.5.1, we must have that $\dim F/_{H^\circ} \leq \dim F/_{L^\circ} \leq c$.

Corollary

Let (E, F) be a dual pair and let G be a $\sigma(F, E)$-dense subspace of F. If $A \subseteq E$ is essentially separable for the dual pair (E, F), it is also essentially separable for the dual pair (E, G).

Proof

This is trivial if $A = \emptyset$. If $A \neq \emptyset$, the proof follows immediately from the theorem when we observe that if H is again the linear span of A, then $G/_{(H^\circ \cap G)}$ is isomorphic to a vector subspace of $F/_{H^\circ}$, where H° is the polar of H in F.

An essentially separable set A in a dual pair (E, F) is required to be contained in a $\sigma(F^*, F)$-separable set. The next result gives us a way of identifying one such containing set.

Theorem 2.1.2

Let (E, F) be a dual pair and let A be a non-empty essentially separable subset of E. If G is the $\sigma(F^*, F)$-closed vector subspace of F^* generated by A, then G is $\sigma(F^*, F)$-separable. If in addition A is $\sigma(E, F)$- bounded, then there is a $\sigma(F^*, F)$-bounded separable subset of G which contains A.

Proof

That G is $\sigma(F^*, F)$-separable is already established in the proof of Theorem 2.1.1.

The second part is trivial if $A = \{0\}$. Otherwise let H be the linear span of A, let $(e_\lambda)_{\lambda\in\Lambda}$ be a basis in $F/_{H^o}(\neq\{0\})$ and let $t : K^{(\Lambda)} \to F/_{H^o}$ be the associated isomorphism. Suppose that A is $\sigma(E, F)$-bounded. Then $t^*(A)$ is $\sigma(K^\Lambda, K^{(\Lambda)})$-bounded where $t^* : G \to K^\Lambda$ is the transpose of t. Now $t^*(A) \subseteq C = \prod_{\lambda\in\Lambda} p_\lambda(t^*(A))$, where p_λ is the projection of K^Λ onto the λth component space. Also, C is a $\sigma(K^\Lambda, K^{(\Lambda)})$ -bounded separable set since $|\Lambda| \leq c$ (Theorem 2.1.1, Theorem 1.4.1). Thus $t^{*-1}(C)$ is a $\sigma(F^*, F)$-bounded separable subset of G which contains A.

The preservation of some properties of sets by a continuous linear mapping is often desirable. For example, the continuous image of a compact set is again compact. The following lemma shows that essentially separable sets have this property which we use often in subsequent sections.

<u>Lemma 2.1.2</u>

Let (E, F) and (G, H) be dual pairs and let $t : E \to G$ be a weakly continuous linear mapping. If $A \subseteq E$ is essentially separable, so also is $t(A)$.

<u>Proof</u>

If $t' : H \to F$ is the transpose of t and $t'^* : F^* \to H^*$ is the algebraic transpose of t' , then $t'^*(x) = t(x)$ for all $x \in E$. Let B be a $\sigma(F^*, F)$-separable set which contains A . Since t'^* is continuous under $\sigma(F^*, F)$ and $\sigma(H^*, H)$, it follows that $t'^*(B)$ is $\sigma(H^*, H)$-separable. But $t(A) = t'^*|_E(A) \subseteq t'^*(B)$ and so $t(A)$ is essentially separable for the dual pair (G, H) .

<u>Lemma 2.1.3</u>

Let (E, F) be a dual pair and let A be a subset of E . If G is a vector subspace of E which contains A , then A is essentially separable for the dual pair (E, F) if and only if it is essentially separable for the dual pair $(G, F/_{G^o})$.

<u>Proof</u>

Let H be the $\sigma(F^*, F)$- closed linear span of A . We note first that $(F/_{G^o})^*$ is (up to isomorphism) the $\sigma(F^*, F)$-closed linear span of G . Since $A \subseteq G$, we have that H is contained in the $\sigma(F^*, F)$-closed linear span of G and so H is also the $\sigma((F/_{G^o})^*, F/_{G^o})$-closed linear span of A . On H the topologies $\sigma(F^*, F)$ and $\sigma((F/_{G^o})^*, F/_{G^o})$ coincide and so H is $\sigma(F^*, F)$-separable if and only if it is $\sigma((F/_{G^o})^*, F/_{G^o})$-separable. The result now follows from the definition and Theorem 2.1.2

Corollary

Let (E, F) be a dual pair and let G be a $\sigma(E, F)$-dense vector subspace of E. A subset A of G is essentially separable for the dual pair (G, F) if and only if it is essentially separable for the dual pair (E, F).

Proof

Since G is $\sigma(E, F)$-dense, $G^{o} = \{0\}$ and so $F/_{G^{o}}$ is topologically isomorphic to F. The result is now an immediate consequence of the lemma.

Let (E, F) be a dual pair and let A be a non-empty $\sigma(E, F)$-bounded set. Let B be the $\sigma(F^{*}, F)$-closed absolutely convex envelope of A, let H be the linear span of A and let L be the linear span of B. Then $(F/_{H^{o}})^{*}$ is the $\sigma(F^{*}, F)$-closure of H and L is a $\sigma(F^{*}, F)$-dense vector subspace of it. It therefore follows that $(F/_{H^{o}}, L)$ is a dual pair and B is $\sigma(L, F/_{H^{o}})$-compact. Consider the seminorm p_B on $F/_{H^{o}}$ defined by

$$p_B(x) = \sup \{ |\langle x, x'\rangle| \ : \ x' \in B\} , \quad x \in F/_{H^{o}} \quad .$$

If $p_B(x) = 0$, then $\langle x, x'\rangle = 0$ for all $x' \in B$. Since B spans L, this then implies that $\langle x, x'\rangle = 0$ for all $x' \in L$ and $(F/_{H^{o}}, L)$ being a dual pair we must have $x = 0$. It therefore follows that p_B is a norm on $F/_{H^{o}}$ and thus there is a topology of the dual pair $(F/_{H^{o}}, L)$ which is normable. Since any metrizable locally convex topology is the Mackey topology of the corresponding dual pair, this normable topology is $\tau(F/_{H^{o}}, L)$ with B as the closed unit ball of the dual L. We shall denote this normed space by $\mathcal{N}(F, A)$ and its completion by $\mathcal{B}(F, A)$ and sometimes refer to them as the normed and the Banach spaces respectively constructed from A.

The next theorem gives us a characterization of essential separability for such a set A in terms of the spaces $\mathcal{N}(F, A)$ and $\mathcal{B}(F, A)$. First we establish the following lemma.

<u>Lemma 2.1.4</u>

If a normed space E has a total subset D with $|D| \leq c$, then $\dim E \leq c$.

<u>Proof</u>

The linear span X of D has cardinality at most c. Since E is metrizable, each element of E is the limit of a sequence in X. It therefore follows that $|E| \leq c^{\aleph_0} = c$. Thus the dimension of E cannot exceed c.

<u>Theorem 2.1.3</u>

Let (E, F) be a dual pair and let A be a non-empty $\sigma(E, F)$-bounded set. The following are equivalent:

(i) A is essentially separable;

(ii) $\dim \mathcal{N}(F, A) \leq c$;

(iii) $\dim \mathcal{B}(F, A) \leq c$.

<u>Proof</u>

The equivalence of (i) and (ii) is given by Theorem 2.1.1

(ii) $\Longrightarrow$ (iii) is immediate from Lemma 2.1.4. above.

(iii) $\Longrightarrow$ (ii) is trivial.

The following theorem gives an interesting characterization of essential separability for the case when $E = K^{(M)}$ and $E^* = K^M$, where K is the scalar field $\mathbb{R}$ or $\mathbb{C}$ and M is any non-empty index set.

Theorem 2.1.4

A non-empty subset A of K^M is essentially separable for the dual pair $(K^M, K^{(M)})$ if and only if there is a subset M_o of M and a family $\{x'_\mu\}_{\mu\in M\setminus M_o}$ of elements of $K^{(M)}$ such that

(i) $|M_o| \leq c$,

(ii) for each $\mu \in M\setminus M_o$, we have $x'_\mu = \sum_{r=1}^{m(\mu)} \lambda(r, \mu)\, e_{\alpha(r, \mu)}$,

where $\alpha(r, \mu) \in M_o$, $\lambda(r, \mu) \in K$ $(r = 1, \ldots, m(\mu))$ and e_ν is the element $(\delta_{\nu\mu})_{\mu\in M}$ of $K^{(M)}$,

(iii) if $x = (\xi_\mu)_{\mu\in M} \in A$, then $\xi_\mu = \langle x, x'_\mu\rangle$ for each $\mu \in M\setminus M_o$.

Proof

Suppose first that the conditions are satisfied. If p_ν is the projection of K^M onto its νth component space then by Theorem 1.4.1 the product $\pi\{p_\mu(A) : \mu \in M_o\}$ is separable. Let $\{(\eta_\mu^{(n)})_{\mu\in M_o} : n \in \mathbb{N}\}$ be an at most countable dense subset of $\pi\{p_\mu(A) : \mu\in M_o\}$. For each $n \in \mathbb{N}$, define $x_n = (\xi_\mu^{(n)})_{\mu\in M}$ by

$$\xi_\mu^{(n)} = \begin{cases} \eta_\mu^{(n)} & \text{if } \mu \in M_o \\ \sum_{r=1}^{m(\mu)} \lambda(r, \mu)\, \eta_{\alpha(r, \mu)}^{(n)} & \text{if } \mu \in M\setminus M_o \ . \end{cases}$$

Then $\langle x_n , x'_\mu\rangle = \xi_\mu^{(n)}$ for all $n \in \mathbb{N}$ and all $\mu \in M\setminus M_o$.

Let $\{\mu(1), \ldots, \mu(s)\}$ be any non-empty finite subset of M and let ϵ be a positive real number. If $\mu(t) \in M_o$ for some t , put $m(\mu(t)) = 1$, $\lambda(1, \mu(t)) = 1$ and $\alpha(1, \mu(t)) = \mu(t)$. Now define σ and N by

$$\sigma = \max \{|\lambda(r, \mu(t))| : r = 1, 2, \ldots, m(\mu(t)) ; t = 1, 2, \ldots, s\} ,$$

$$N = \max \{m(\mu(t)) : t = 1, 2, \ldots, s\} .$$

For any $x = (\xi_\mu)_{\mu\epsilon M} \epsilon A$, we have $y = (\xi_\mu)_{\mu\epsilon M_o} \in \Pi\{p_\mu(A) : \mu\epsilon M_o\}$

and so there is a positive integer n such that

$$|\xi_{\alpha(r, \mu(t))} - \eta^{(n)}_{\alpha(r, \mu(t))}| \le \frac{\epsilon}{N(\sigma+1)} \quad (r = 1, 2, \ldots, m(\mu(t)) ; t = 1, \ldots, s) .$$

Then we have

$$|\xi_{\mu(t)} - \xi^{(n)}_{\mu(t)}| = |\sum_{r=1}^{m(\mu(t))} \lambda(r,\mu(t))\, \xi_{\alpha(r,\mu(t))}$$

$$- \sum_{r=1}^{m(\mu(t))} \lambda(r,\mu(t))\, \eta^{(n)}_{\alpha(r, (t))}| \le \sum_{r=1}^{m(\mu(t))} |\lambda(r,\mu(t))||\xi_{\alpha(r,\mu(t))} - \eta^{(n)}_{\alpha(r,\mu(t))}|$$

$$\le N\sigma \cdot \frac{\epsilon}{N(\sigma+1)} \le \epsilon .$$

It therefore follows that A is contained in the closure of $\{x_n : n \epsilon \mathbb{N}\}$ and so it is essentially separable.

Suppose conversely that A is essentially separable for the dual pair $(K^M , K^{(M)})$. If $A = \{0\}$ the conditions are clearly satisfied with any non-empty subset M_o of M with $|M_o| \le c$ and $x'_\mu = 0$ $(\mu\epsilon M\backslash M_o)$. Otherwise let H be the linear span of A and let $(f_\phi)_{\phi\epsilon\Phi}$ be a basis in $E = K^{(M)}/_{H^o}$. Then by Theorem 2.1.1, we must have $|\Phi| \le c$. Now let $q : K^{(M)} \to E$ be the quotient map. For each $\phi \epsilon \Phi$, choose $e_{\mu(r, \phi)}$ and scalars $\gamma(r, \phi)$ $(r = 1, \ldots, n(\phi))$ such that $f_\phi = q(\sum_{r=1}^{n(\phi)} \gamma(r, \phi)\, e_{\mu(r,\phi)})$. We now put $M_o = \{\mu(r,\phi) : r = 1, 2, \ldots, n(\phi) ; \phi \epsilon \Phi\}$. Then we have

$|M_o| \leq c \aleph_0 = c$ since $|\Phi| \leq c$ and for each $\phi \in \Phi$ there are finitely many r's .

If $M = M_o$, then there is nothing to prove. Otherwise if $\nu \in M \setminus M_o$, choose $\phi(s,\nu) \in \Phi$, $\beta(s,\nu) \in K$ $(s = 1, \ldots, m(\nu))$ such that $q(e_\nu) = \sum_{s=1}^{m(\nu)} \beta(s,\nu) f_{\phi(s,\nu)}$. Then we have

$$q(e_\nu) = q\Big(\sum_{s=1}^{m(\nu)} \sum_{r=1}^{n(\phi(s,\nu))} \beta(s,\nu)\,\gamma(r,\phi(s,\nu))\, e_{\mu(r,\phi(s,\nu))}\Big) .$$

Now if $x'_\nu = \sum_{s=1}^{m(\nu)} \sum_{r=1}^{n(\phi(s,\nu))} \beta(s,\nu)\,\gamma(r,\phi(s,\nu))\, e_{\mu(r,\phi(s,\nu))}$, then for all $x = (\xi_\mu)_{\mu \in M}$ in H (in fact for all x in the closure of H) we have $\xi_\nu = \langle x, e_\nu \rangle = \langle x, x'_\nu \rangle$. This holds in particular for all $x \in A$ and hence we have the required set M_o and the family $\{x'_\mu\}_{\mu \in M \setminus M_o}$.

__Example 2.1.1__

If A is a subset of K^M , the support of A (supp A) is defined to be $\{\mu_o \in M : \exists\ x = (\xi_\mu)_{\mu \in M} \in A$ such that $\xi_{\mu_o} \neq 0\}$.

Any subset A of K^M with $|\text{supp } A| \leq c$ is essentially separable for the dual pair $(K^M, K^{(M)})$. If $A = \emptyset$ or $A = \{0\}$ this is trivial. Otherwise in Theorem 2.1.4 above we may take $M_o = \text{supp } A$ and $x'_\mu = 0$ for all $\mu \in M \setminus M_o$.

We note however that the condition that the cardinality of the support be at most c is not necessary for essential separability. For example if A is a subset of K^M with $|\text{supp } A| \leq c$, we can define $B \subseteq K^M$ by $B = A \cup \{(1)_{\mu \in M}\}$. Then B is essentially separable, for if C is a $\sigma(K^M, K^{(M)})$-separable set which contains

A , then $C \cup \{(1)_{\mu \in M}\}$ is a $\sigma(K^M, K^{(M)})$-separable set which contains B . Now $\operatorname{supp} B = M$ and we may choose M to have cardinality strictly greater than c . In applying Theorem 2.1.4 to B when $\operatorname{supp} A \nsubseteq M$ we can choose $\mu_o \in M\backslash \operatorname{supp} A$ and take $M_o = (\operatorname{supp} A) \cup \{\mu_o\}$ and $x'_\mu = e_{\mu_o}$ for each $\mu \in M\backslash M_o$.

The following results give some useful topological properties associated with essential separability.

<u>Theorem 2.1.5</u>

If A is essentially separable for the dual pair (E, F) , then $\sigma(E, F)|_A$ has a base consisting of at most c sets.

<u>Proof</u>

If A is empty, then the result is trivially true. Suppose A is non-empty and let H be the linear span of A and let G be the $\sigma(F^*, F)$-closed linear span of A . Then by Theorem 2.1.1, if H^o is the polar of H in F , we have that the dimension and consequently the cardinality of F/H^o are at most c . It follows also from Theorem 2.1.2 that G is $\sigma(F^*, F)$-separable.

Let Φ be the set of all non-empty finite subsets of F/H^o and $\{x_n : n \in \mathbb{N}\}$ be an at most countable $\sigma(F^*, F)$-dense subset of G . Then $|\Phi| \le c$ and if we define $\mathfrak{X}$ by

$$\mathfrak{X} = \{\{x \in A : |\langle x - x_n, x'\rangle| < 1 ;\ x' \in \phi\} : \phi \in \Phi\} ,$$

then $|\mathfrak{X}| \le c$. Since $\sigma(F^*, F)$, $\sigma(G, F/_{H^o})$ and $\sigma(E, F)$ all coincide on A , each element of $\mathfrak{X}$ is $\sigma(E, F)|_A$ -open. We shall show next that $\mathfrak{X}$ is a base for $\sigma(E, F)|_A$.

Let $y \in A$ and let U be any $\sigma(E, F)|_A$ -neighbourhood of y . There exists $\phi_o \in \Phi$ such that $V \subseteq U$, where

$$V = \{x \in A : |\langle x-y , x'\rangle| < 1 ,\ x' \in \phi_o\} .$$

Since $\{x_n : n \in \mathbb{N}\}$ is a $\sigma(F^*,F)$-dense subset of G (and therefore $\sigma(G, F/_{H^0})$-dense) there exists $n_0 \in \mathbb{N}$ such that

$$x_{n_0} \in \{x \in G : |\langle x - y, x'\rangle| < 1, x' \in 2\phi_0\} .$$

We now define $W \in \mathfrak{X}$ by

$$W = \{x \in A : |\langle x - x_{n_0}, x'\rangle| < 1, x' \in 2\phi_0\} .$$

It is then clear that $y \in W$. Also, if x is any element of W, then $x \in A$ and for any $x' \in \phi_0$, we have

$$\begin{aligned} |\langle x - y, x'\rangle| &= \tfrac{1}{2}|\langle x - y, 2x'\rangle| \\ &\le \tfrac{1}{2}|\langle x - x_{n_0}, 2x'\rangle| + \tfrac{1}{2}|\langle x_{n_0} - y, 2x'\rangle| \\ &< \tfrac{1}{2} + \tfrac{1}{2} = 1 , \quad \text{since } 2x' \in 2\phi_0 . \end{aligned}$$

Hence $x \in V$ and so $W \subseteq V \subseteq U$. Thus $\mathfrak{X}$ is a base for $\sigma(E, F)|_A$ and this establishes the result.

<u>Corollary</u>

If A is essentially separable for the dual pair (E, F), then A has a $\sigma(E, F)$-dense subset of cardinality at most c.

<u>Proof</u>

This follows immediately from the fact that any topological space has a dense subset of cardinality at most that of a given base for its topology.

It is clear from Theorem 2.1.5 that each point in an essentially separable set has a base of neighbourhoods for the topology induced by the weak topology consisting of at most c sets. It will be shown later (Theorem 2.1.7) that for a non-empty absolutely convex $\sigma(E, F)$-

bounded set A to be essentially separable for the dual pair (E, F) it is in fact necessary and sufficient that zero has a base of neighbourhoods for $\sigma(E, F)|_A$ consisting of at most c sets. An example will be given later too to show that the converse of the Corollary to Theorem 2.1.5 is not true in general.

The next result describes precisely the situation when the result of Theorem 2.1.5 holds for an absolutely convex set. It is an analogue of [25, Proposition 1.3].

<u>Theorem 2.1.6</u>

Let (E, F) be a dual pair and let A be a non-empty absolutely convex subset of E. Then $\sigma(E, F)|_A$ has a base consisting of at most c sets if and only if

(i) 0 has a base of neighbourhoods for $\sigma(E, F)|_A$ consisting of at most c sets,

(ii) A has a $\sigma(E, F)$-dense subset of cardinality at most c.

<u>Proof</u>

Since $0 \in A$, it is clear that the conditions are necessary.

Suppose conversely that (i) and (ii) are satisfied. Let Φ be the set of all non-empty finite subsets of F. Then there exists $\{\phi_\lambda : \lambda \in \Lambda\} \subseteq \Phi$ with $|\Lambda| \leq c$ such that the family of sets of the form $\{x \in A : |\langle x, x'\rangle| < 1, x' \in \phi_\lambda\}$ $(\lambda \in \Lambda)$ is a base of neighbourhoods of 0 for $\sigma(E, F)|_A$. Let $\{x_\mu : \mu \in M\}$ be a $\sigma(E, F)$-dense subset of A with $|M| \leq c$ and put

$$\mathcal{X} = \{\{x \in A : |\langle x - x_\mu, x'\rangle| < 1, x' \in \phi_\lambda\} : \lambda \in \Lambda, \ \mu \in M\} .$$

Then we have that $|\mathcal{X}| \leq c$. We show next that $\mathcal{X}$ is a base for $\sigma(E, F)|_A$.

Let $y \in A$ and let V be any $\sigma(E, F)|_A$ -neighbourhood of y . Then there exists $\phi \in \Phi$ such that

$$\{x \in A : |\langle x - y, x'\rangle| < 1, x' \in \phi\} \subseteq V \quad . \qquad (*)$$

Choose $\lambda_o \in \Lambda$ such that

$$\{x \in A : |\langle x, x'\rangle| < 1, x' \in \phi_{\lambda_o}\} \subseteq \{x \in A : |\langle x, x'\rangle| < 1, x' \in 4\,\phi\} \quad . (**)$$

Since $\{x_\mu : \mu \in M\}$ is a $\sigma(E,F)$-dense subset of A , there exists $\mu_o \in M$ such that $x_{\mu_o} \in \{x \in A : |\langle x - y, x'\rangle| < 1, x' \in \phi_{\lambda_o}\}$. Let $U = \{x \in A : |\langle x - x_{\mu_o}, x'\rangle| < 1, x' \in \phi_{\lambda_o}\}$. Then $U \in \mathcal{Y}$ and U is a $\sigma(E, F)|_A$ -neighbourhood of y . Besides, for all $x \in U$, we have $\frac{1}{2}(x - x_{\mu_o})$ and $\frac{1}{2}(y - x_{\mu_o})$ belong to A since A is absolutely convex. Therefore by (**) above, for all $x \in U$ and $x' \in \phi$, we have

$$|\langle x - y, x'\rangle| \le \tfrac{1}{2}(|\langle\tfrac{1}{2}(x - x_{\mu_o}), 4\,x'\rangle| + |\langle\tfrac{1}{2}(x_{\mu_o} - y), 4\,x'\rangle|) < 1 \; .$$

It follows now from (*) that $U \subseteq V$ and hence $\mathcal{Y}$ is a base for $\sigma(E, F)|_A$ as required.

The following includes a partial converse of Theorem 2.1.5

Theorem 2.1.7

Let (E, F) be a dual pair and let A be a non-empty absolutely convex $\sigma(E, F)$-bounded set. Then A is essentially separable if and only if 0 has a base of neighbourhoods for $\sigma(E, F)|_A$ consisting of at most c sets.

Proof

The necessity of the condition is an immediate consequence of Theorem 2.1.5

Suppose the condition is satisfied. Then there exists a set $\{\phi_\lambda : \lambda \in \Lambda\}$ of non-empty finite subsets of F, such that $|\Lambda| \le c$ and $\{\{x \in A : |\langle x, x'\rangle| < 1, x' \in \phi_\lambda\} : \lambda \in \Lambda\}$ is a base of neighbourhoods of 0 for $\sigma(E, F)|_A$. The $\sigma(F^*, F)$-closure B of A is an absolutely convex $\sigma(F^*, F)$-compact set. We now show that $\{\{x \in B : |\langle x, x'\rangle| < 1, \forall x' \in \phi_\lambda\} : \lambda \in \Lambda\}$ is a base of neighbourhoods of 0 for $\sigma(F^*, F)|_B$. Let ϕ be any non-empty finite subset of F. Since A is a $\sigma(F^*, F)$-dense subset of B, $U = \{x \in A : |\langle x, x'\rangle| < 1, \forall x' \in \phi\}$ is $\sigma(F^*, F)$-dense in $\{x \in B : |\langle x, x'\rangle| < 1, \forall x' \in \phi\}$ and since U is a $\sigma(E, F)|_A$-neighbourhood of 0, there exists ϕ_{λ_0} such that $V = \{x \in A : |\langle x, x'\rangle| < 1, \forall x' \in \phi_{\lambda_0}\} \subseteq U$. Then $\{x \in B : |\langle x, x'\rangle| < 1, \forall x' \in \phi_{\lambda_0}\} \subseteq \overline{V} \subseteq \overline{U} =$ $\{x \in B : |\langle x, x'\rangle| \le 1, \forall x' \in \phi\}$, where $-$ denotes $\sigma(F^*, F)|_B$-closure. This establishes the assertion.

Let z be a non-zero element of B. Since $\sigma(F^*, F)|_B$ is a separated topology, there exists $\lambda_0 \in \Lambda$ such that $|\langle z, x'\rangle| \ge 1$ for some $x' \in \phi_{\lambda_0}$. It therefore follows that if $T = \cup \{\phi_\lambda : \lambda \in \Lambda\}$, then T separates the elements of B and so the set of equivalence classes in $\mathcal{N}(F, A)$ of the elements of T is a total subset of $\mathcal{N}(F, A)$. Besides, since $|\Lambda| \le c$, we must have $|T| \le c$. From Lemma 2.1.4, it follows that $\mathcal{N}(F, A)$ has dimension at most c and consequently by Theorem 2.1.3, we have that A is essentially separable.

<u>Corollary 1</u>

Let E be a separated locally convex space and let A be a non-empty equicontinuous essentially separable subset of E'. If H is any vector subspace of the completion G of E which contains E, then A is essentially separable for the dual pair (E', H) .

Proof

By Lemma 2.1.1 and the fact that the absolutely convex envelope of an equicontinuous set is also equicontinuous it is enough to establish the result when A is absolutely convex. We note also that an equicontinuous subset of E' is necessarily $\sigma(E', E)$-bounded.

From the discussion preceding Proposition 7 of [43, Chapter VI], we know that the dual of G is, up to isomorphism E' . Also by [28, §21, 6(2)], both $\sigma(E', G)$ and $\sigma(E', E)$ coincide on A . Since $E \subseteq H \subseteq G$, it follows that $\sigma(E', H)|_A = \sigma(E', E)|_A$. Applying the theorem once more, we get that A is essentially separable for the dual pair (E', H) .

Corollary 2

Let E be a separated locally convex space and let G be a $\beta(E, E')$-dense vector subspace of E . If A is a $\sigma(E', E)$-bounded set which is essentially separable for the dual pair (E', G) , then A is also essentially separable for the dual pair (E', E) .

Proof

Let F be the $\beta(E, E')$-completion of E and let F' be the dual of F . Since G is $\beta(E, E')$-dense in E , we have that (G, F') is a dual pair and that the completion of G under $\beta(E, E')|_G$ is F . Since A is equicontinuous for $\beta(E, E')|_G$, Corollary 1 above and the Corollary to Lemma 2.1.3 show that A is essentially separable for the dual pair (F', F) . From the Corollary to Theorem 2.1.1, it follows that A is essentially separable for the dual pair (F', E) . Applying the Corollary to Lemma 2.1.3 again gives the result.

We now give an example to illustrate the fact that the Corollary to Theorem 2.1.5 does not have a converse similar to Theorem 2.1.7.

Example 2.1.2

Consider the dual pair $(\ell_\infty'', \ell_\infty')$. It is known (see for example [51, Lemma 4.61B]) that if E is a normed space its closed unit ball is $\sigma(E'', E')$-dense in the closed unit ball of E'' . Hence the closed unit ball A of ℓ_∞ is a $\sigma(\ell_\infty'', \ell_\infty')$-dense subset of the closed unit ball B of ℓ_∞'' and $|A| = c$. It is clear from the definitions that $\mathcal{N}(\ell_\infty', A) = \mathcal{B}(\ell_\infty', A) \simeq \ell_\infty' \simeq \mathcal{N}(\ell_\infty', B) = \mathcal{B}(\ell_\infty', B)$.

From Note 1.8(a) of [59], we know that there are at least 2^c distinct bounded finitely additive measures on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, where $\mathcal{P}(\mathbb{N})$ is the set of all subsets of $\mathbb{N}$. By Theorem 2.3 of the same paper, ℓ_∞' may be represented as the set of all bounded finitely additive measures on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. It then follows that the dimension of ℓ_∞' is at least 2^c. For if $\dim \ell_\infty' = \alpha$ where $\alpha < 2^c$, then ℓ_∞' would have at most $c \cdot \alpha = \max(c, \alpha) < 2^c$ distinct elements which is false. Since $\dim \ell_\infty = c$, we have $\dim \ell_\infty' \le \dim \ell_\infty^* = c^c = 2^c$. Hence $\dim \ell_\infty' = 2^c$, and consequently the dimension of $\mathcal{N}(\ell_\infty', A) = \dim \mathcal{B}(\ell_\infty', A)$ is $2^c > c$. Thus neither A nor B is essentially separable for the dual pair $(\ell_\infty'', \ell_\infty')$ even though $|A| = c$.

Thus the fact that a set contained in E in a dual pair (E, F) has a $\sigma(E, F)$-dense subset of cardinality at most c does not necessarily imply that it is essentially separable for the dual pair.

However, considering A in the above case as a subset of ℓ_∞ in the dual pair (ℓ_∞, ℓ_1) , we have that A is essentially separable since $\mathcal{N}(\ell_1, A) = \mathcal{B}(\ell_1, A) \simeq \ell_1$ and $\dim \ell_1 = c$. This example also shows the need for specifying the dual pair for which a given set is said to be essentially separable.

We recall the following definition of the Schauder dimension of a Banach space given in [16]. A subset M of a Banach space X is said to be strongly linearly independent if it is not contained in the closed linear span of any proper subset. A maximal strongly linearly independent subset of X is called an extended Schauder basis for X. Any two extended Schauder bases in the same space X have the same cardinality which is therefore called the Schauder dimension of X.

It is known (Corollary to Theorem 2.1.5 and Lemma 2.1.1) that every subset of an essentially separable set in a locally convex space E has a $\sigma(E, E')$-dense subset of cardinality at most c. We give next a result which combines this property of essentially separable sets with the concept of Schauder dimension for Banach spaces.

Theorem 2.1.8

Let E be a Banach space and suppose that every subset of the closed unit ball of E' has a $\sigma(E', E)$-dense subset of cardinality at most c. If E has a Schauder dimension, then the dimension of E is at most c.

Proof

Let $\{x_\lambda : \lambda \in \Lambda\}$ be a maximal strongly linearly independent subset of E and let $\{x'_\lambda : \lambda \in \Lambda\}$ be a subset of E' such that $\langle x_\mu, x'_\lambda \rangle = \delta_{\lambda\mu}$, for all $\lambda, \mu \in \Lambda$. Let $A = \{\|x'_\lambda\|^{-1}x'_\lambda : \lambda \in \Lambda\}$. Then A is a subset of the closed unit ball of E'. Also if $\lambda \neq \mu$, then we have $\langle x_\lambda, \|x'_\lambda\|^{-1}x'_\lambda - \|x'_\mu\|^{-1}x'_\mu \rangle = \|x'_\lambda\|^{-1}$ and so A has no proper $\sigma(E', E)$-dense subset. Otherwise, if $\{\|x'_\gamma\|^{-1}x'_\gamma : \gamma \in \Gamma\}$ were a proper $\sigma(E', E)$-dense subset of A, given $\mu \in \Lambda\backslash\Gamma$, we would be able to find $\nu \in \Gamma$ such that $|\langle x_\mu, \|x'_\mu\|^{-1}x'_\mu - \|x'_\nu\|^{-1}x'_\nu \rangle| < \|x_\mu\|^{-1}$ which is false. By the hypothesis we must have $|\Lambda| \leq c$ and from [16, Proposition 1], the set $\{x_\lambda : \lambda \in \Lambda\}$ is total in E. Applying Lemma 2.1.4, we have the desired result.

We end this section with two lemmas which we shall need later in our subsequent discussions. The second of these shows that essential separability is preserved in forming products in the same way as separability is preserved (Theorem 1.4.1).

Let E, F be complex vector spaces forming a dual pair (E, F). The set of vectors in E can be made into a real vector space $E_{\mathbb{R}}$ by simply restricting scalar multiplication to real scalars. The set F_r of real linear forms $x \to \mathrm{Re}\,\langle x, y\rangle$ $(y \in F)$ is then a real vector space such that $(E_{\mathbb{R}}, F_r)$ is a dual pair (see for example [28, §16, 3, §21, 11]).

Lemma 2.1.5

In the above notation, a subset A of E is essentially separable for the dual pair (E, F) if and only if it is essentially separable for the dual pair $(E_{\mathbb{R}}, F_r)$.

Proof

We apply the same construction to the dual pair (F^*, F) to get $((F^*)_{\mathbb{R}}, F_r)$. Since $\sigma(F^*, F)$ and $\sigma((F^*)_{\mathbb{R}}, F_r)$ coincide [28, §21, 11(2)], A is contained in a $\sigma(F^*, F)$-separable set if and only if it is contained in a $\sigma((F^*)_{\mathbb{R}}, F_r)$-separable set. Since F^* is complete under $\sigma(F^*, F)$, we must have $(F^*)_{\mathbb{R}} = (F_r)^*$. The result now follows from the definition of essential separability.

Lemma 2.1.6

Let (E_λ, F_λ) $(\lambda \in \Lambda)$ be dual pairs and for each $\lambda \in \Lambda$, let A_λ be a non-empty subset of E_λ. Then $\prod_{\lambda\in\Lambda} A_\lambda$ is essentially separable for the dual pair $(\prod_{\lambda\in\Lambda} E_\lambda, \sum_{\lambda\in\Lambda} F_\lambda)$ if and only if

(i) each A_λ is essentially separable for the dual pair (E_λ, F_λ), and

(ii) $|\{\lambda : |A_\lambda| \geq 2\}| \leq c$.

<u>Proof</u>

Suppose the conditions are satisfied. If $|A_\lambda| \geq 2$, choose a $\sigma(F_\lambda{}^*, F_\lambda)$-separable set B_λ such that $A_\lambda \subseteq B_\lambda$. If $|A_\lambda| = 1$, let $B_\lambda = A_\lambda$. Then by Theorem 1.4.1, $\prod_{\lambda\in\Lambda} B_\lambda$ is a $\sigma(\prod_{\lambda\in\Lambda} F^*{}_\lambda, \sum_{\lambda\in\Lambda} F_\lambda)$-separable set which contains $\prod_{\lambda\in\Lambda} A_\lambda$. Since $(\sum_{\lambda\in\Lambda} F_\lambda)^* = \prod_{\lambda\in\Lambda} F_\lambda{}^*$, it follows that $\prod_{\lambda\in\Lambda} A_\lambda$ is essentially separable as required.

Applying Lemma 2.1.2 to the canonical projections $p_\lambda : \prod_{\lambda\in\Lambda} E_\lambda \to E_\lambda$ $(\lambda\in\Lambda)$ establishes the necessity of condition (1).

Suppose that each E_λ is non-zero and for each $\lambda\in\Lambda$, let M_λ be a set such that $|M_\lambda| = \dim F_\lambda$ and $M_{\lambda_1} \cap M_{\lambda_2} = \emptyset$ if $\lambda_1 \neq \lambda_2$. In the usual way, we may identify F_λ with $K^{(M_\lambda)}$, $F_\lambda{}^*$ with K^{M_λ} and E_λ with a vector subspace of K^{M_λ}. Suppose $\prod_{\lambda\in\Lambda} A_\lambda$ is essentially separable for the dual pair $(\prod_{\lambda\in\Lambda} E_\lambda, \sum_{\lambda\in\Lambda} F_\lambda)$. With the above identifications, we have by the corollary to Lemma 2.1.3 and Lemma 2.1.2 that $\prod_{\lambda\in\Lambda} A_\lambda$ is essentially separable for the dual pair $(K^M, K^{(M)})$ where $M = \cup\{M_\lambda : \lambda\in\Lambda\}$.

Now apply Theorem 2.1.4 to $\prod_{\lambda\in\Lambda} A_\lambda$. There is a non-empty subset Γ of Λ such that $|\Gamma| \leq c$ and $M_o \subseteq \cup\{M_\lambda : \lambda\in\Gamma\}$. Suppose there exists $\lambda_o\in\Lambda\setminus\Gamma$ such that $|A_{\lambda_o}| \geq 2$. For each $\lambda\in\Lambda\setminus\{\lambda_o\}$ choose $a_\lambda \in A_\lambda$ and let b_{λ_o} , c_{λ_o} be distinct elements of A_{λ_o} . Then if

$$\xi_\lambda = \begin{cases} a_\lambda & \lambda \neq \lambda_o \\ b_{\lambda_o} & \lambda = \lambda_o \end{cases} \qquad \eta_\lambda = \begin{cases} a_\lambda & \lambda \neq \lambda_o \\ c_{\lambda_o} & \lambda = \lambda_o \end{cases}$$

we have that $x = (\xi_\lambda)_{\lambda\in\Lambda}$ and $y = (\eta_\lambda)_{\lambda\in\Lambda}$ are distinct elements of

$\prod_{\lambda \in \Lambda} A_\lambda$ whose components in K^M are the same over M_0. Thus from the properties of the elements $x'_\mu \in K^{(M)}$ $(\mu \in M \backslash M_0)$ it follows that x and y are the same, which is a contradiction. Thus $|A_\lambda| = 1$ if $\lambda \in \Lambda \backslash \Gamma$. This establishes the necessity of condition (ii) when each $E_\lambda \neq \{0\}$. The result is trivial if some $E_\lambda = \{0\}$; otherwise we can reduce the problem to the non-zero case by observing that $\prod_{\lambda \in \Lambda} E_\lambda$ is topologically isomorphic to $\prod_{\substack{\lambda \in \Lambda \\ E_\lambda \neq \{0\}}} E_\lambda$.

2.2 Weak Compactness and Essential Separability

In this section, we shall identify some particular essentially separable sets and bring out a link with ideas associated with weak compactness and weak relative compactness.

In a dual pair (E, F) an arbitrary subset A of E of cardinality at most c need not be essentially separable. For example, the closed unit ball A of ℓ_∞ of Example 2.1.2 has cardinality c but it is not essentially separable for the dual pair $(\ell''_\infty, \ell'_\infty)$. In the situation when the subset is in addition $\sigma(E, F)$-compact we have the following result.

Theorem 2.2.1

Let (E, F) be a dual pair and let A be a $\sigma(E, F)$-compact set with cardinality at most c. Then A is essentially separable for the dual pair (E, F).

Proof

The result is trivial if A is empty. Also by Lemma 2.1.5 and [28, §21, 11(1)], it is enough to establish the result when E is real.

Suppose that A is non-empty and for each pair of distinct elements x_1, x_2 of A choose $y \in F$ such that $\langle x_1, y\rangle \neq \langle x_2, y\rangle$. Consider $C(A)$ the algebra of all continuous real-valued functions on A. Let $\mathcal{A}$ be the subalgebra of $C(A)$ generated by the set X consisting of the restrictions to A of all the y chosen above together with the unit function. Then by the Stone-Weierstrass Theorem [10, Chapter XIII, Theorem 3.3] we have that $\mathcal{A}$ is dense in $C(A)$ for the topology of uniform convergence on A. By [10, Chapter XIII, 3.1] $\mathcal{A}$ consists of all functions of the form $p(f_1, \ldots, f_n)$, where $f_1, \ldots, f_n$ belong to X and p ranges over all polynomials in $n \geq 1$ indeterminates with no constant terms. Since there are c such polynomials and at most c functions from which to fill the indeterminates, it follows that $|\mathcal{A}| = c$. As the topology of uniform convergence on A is a norm topology, every element of $C(A)$ is the limit of a sequence in $\mathcal{A}$ since $\mathcal{A}$ is dense in $C(A)$. It therefore follows that $|C(A)| = c$ and consequently the dimension of $C(A)$ is at most c. This then implies that the vector subspace $L = \{y|_A : y \in F\}$ also has dimension at most c.

Let $\{y_\lambda : \lambda\in\Lambda\}$ be a subset of F such that $(y_\lambda|_A)_{\lambda\in\Lambda}$ is a basis for L (so that $|\Lambda| \leq c$). Let H be the linear span of A. We show that $(\overline{y}_\lambda)_{\lambda\in\Lambda}$ is a basis in $F/_{H^0}$, where $\overline{y}_\lambda$ indicates the equivalence class of y_λ. Suppose that

$\sum_{r=1}^{n} \alpha_r \overline{y}_{\lambda_r} = 0 \in F/_{H^0}$. Then $\langle \sum_{r=1}^{n} \alpha_r y_{\lambda_r}, x\rangle = 0$ for all $x \in H$.

Hence $(\sum_{r=1}^{n} \alpha_r y_{\lambda_r})|_A = 0$, i.e. $\sum_{r=1}^{n} \alpha_r y_{\lambda_r}|_A = 0$. Since $(y_\lambda|_A)_{\lambda\in\Lambda}$ is a basis, we must have $\alpha_r = 0$, $(r = 1, \ldots, n)$ and so the $\overline{y}_\lambda$ are linearly independent. Now let $\overline{y} \in F/_{H^0}$. We have $y_1 \in \overline{y}$

if and only if $\langle y_1, x\rangle = \langle y, x\rangle$ for all $x \in A$. Now

$y|_A = \sum_{t=1}^{s} \beta_t y_{\lambda_t}|_A$ for some $\lambda_t \in \Lambda (t = 1, \ldots, s)$ and some scalars

$\beta_t (t = 1, \ldots, s)$ and consequently we have $\bar{y} = \sum_{t=1}^{s} \beta_t \bar{y}_{\lambda_t}$. Thus

the $\bar{y}_\lambda$ $(\lambda \in \Lambda)$ span $F/_{H^\circ}$ and since we have seen that they are linearly independent, $(\bar{y}_\lambda)_{\lambda \in \Lambda}$ is a basis in $F/_{H^\circ}$. Hence $\dim F/_{H^\circ} \leq c$ and from Theorem 2.1.1 it follows that A is essentially separable.

<u>Corollary 1</u>

Let (E, F) be a dual pair and let A be a subset of E such that

(i) $|A| \leq c$,

(ii) there exist $\sigma(E, F)$-compact sets A_n $(n \in \mathbb{N})$ such that

$$A = \bigcup_{n=1}^{\infty} A_n .$$

Then A is essentially separable for the dual pair (E, F).

<u>Proof</u>

Clearly, for each $n \in \mathbb{N}$, the cardinality of A_n is at most c and so by Theorem 2.2.1 each A_n is essentially separable. From Lemma 2.1.1, we have that $\bigcup_{n=1}^{\infty} A_n$ is essentially separable.

<u>Corollary 2</u>

Let E be a normed space and let B be the closed unit ball of E'. If $|B| = c$ then $\dim E \leq c$.

Proof

We note that for the dual pair (E', E), B is a $\sigma(E', E)$-compact set with cardinality (at most) c. It therefore follows from the theorem that B is essentially separable for the dual pair (E', E). Consequently by Theorem 2.1.3 we have that $\dim \mathcal{N}(E, B) \leq c$, where as in Section 2.1 $\mathcal{N}(E, B)$ is the normed space constructed from B. It is clear that $\mathcal{N}(E, B) \simeq E$ and so $\dim E \leq c$.

As an application of Theorem 2.2.1, we have the following result which gives us further conditions under which a set of cardinality at most c is essentially separable.

Theorem 2.2.2

Let E be a separated locally convex space whose topology is defined by at most c seminorms and let F be the completion of E. Then each subset of E which is $\sigma(F, E')$-relatively compact and whose cardinality is at most c is essentially separable for the dual pair (E, E').

Proof

We note first that by [28, §18, 4(2)], the topology of F too is defined by at most c seminorms and also from the discussion preceding Proposition 7 of Chapter VI of [43], the dual of F is, up to isomorphism, E'. It is therefore enough to establish the result when E is complete. Using [43, Chapter V, Proposition 16 and the Corollary to Proposition 19], we may regard E as a subspace of a product $\pi\{E_\lambda : \lambda\in\Lambda\}$ of Banach spaces $E_\lambda (\lambda\in\Lambda)$ where $|\Lambda| \leq c$.

For each $\lambda \in \Lambda$, let p_λ be the canonical projection of the product onto E_λ. Now let A be a non-empty $\sigma(E, E')$-relatively compact set with $|A| \leq c$. For each $\lambda \in \Lambda$, $p_\lambda(A)$ is $\sigma(E_\lambda, E_\lambda')$-

relatively compact since p_λ is continuous for each $\lambda \in \Lambda$. Since $|A| \leq c$, we have that $|p_\lambda(A)| \leq c$, and since every element of the $\sigma(E_\lambda, E'_\lambda)$-closure of $p_\lambda(A)$ is the $\sigma(E_\lambda, E'_\lambda)$-limit of a sequence in $p_\lambda(A)$ [28, §24, 1(7)] it follows that the $\sigma(E_\lambda, E'_\lambda)$-closure of $p_\lambda(A)$ has cardinality at most c. Hence for each $\lambda \in \Lambda$, we have that $p_\lambda(A)$ is essentially separable for the dual pair (E_λ, E'_λ), by Theorem 2.2.1

By Lemma 2.1.6, we have that $\prod_{\lambda\in\Lambda} p_\lambda(A)$ is essentially separable for the dual pair $(\prod_{\lambda\in\Lambda} E_\lambda, \sum_{\lambda\in\Lambda} E'_\lambda)$ since $|\Lambda| \leq c$. Thus since $A \subseteq \prod_{\lambda\in\Lambda} p_\lambda(A)$, it follows that A is essentially separable for the dual pair $(\prod_{\lambda\in\Lambda} E_\lambda, \sum_{\lambda\in\Lambda} E'_\lambda)$. We apply Lemma 2.1.3 to conclude that A is essentially separable for the dual pair (E, E') and the proof is complete.

Remarks

(i) As we have in Example 2.1.2, let A be the closed unit ball of $\ell_\infty \subseteq \ell''_\infty$. Then $|A| = c$ and the $\sigma(\ell''_\infty, \ell'_\infty)$-closure of A is $\sigma(\ell''_\infty, \ell'_\infty)$-compact. But A is not essentially separable for the dual pair $(\ell''_\infty, \ell'_\infty)$. Since A satisfies the conditions of Theorem 2.2.2, we deduce that no topology of the dual pair $(\ell''_\infty, \ell'_\infty)$ can be defined by at most c seminorms.

(ii) We note also that Theorem 2.2.2 does not hold for an arbitrary subset with cardinality c. To see this we take $E = \ell_\infty$, $F = \ell'_\infty$ and $A = E$ so that $|A| = |\ell_\infty| = c$. The norm topology on ℓ_∞ is a topology of the dual pair $(\ell_\infty, \ell'_\infty)$ which is defined by 1 $(<c)$ seminorm. However ℓ_∞ is not essentially separable for the dual pair $(\ell_\infty, \ell'_\infty)$ since $\ell'_\infty/\ell^o_\infty \simeq \ell'_\infty$ and $\dim \ell'_\infty = 2^c$ (Theorem 2.1.1 and Example 2.1.2).

For the corollary to Theorem 2.2.2, we need the following lemma which is stated by Valdivia in [55, (b)].

Lemma 2.2.1

Let E be a locally convex space with dual E' which is the union of the infinite family $\{D_\lambda : \lambda \in \Lambda\}$ of $\sigma(E', E)$-compact sets D_λ with $|\Lambda| \leq \alpha$. If A is an arbitrary subset of E then every weak closure point x_o of A is a weak closure point of a subset M of A with $|M| \leq \alpha$.

Corollary (to Theorem 2.2.2)

Let E be a separated locally convex space whose topology is defined by at most c seminorms and let F be the completion of E. Suppose that B is a subset of E which is $\sigma(F, E')$-relatively compact. If x is any element of the $\sigma(F, E')$-closure of B, then there is an essentially separable subset A of B such that x is in the $\sigma(F, E')$-closure of A.

Proof

Let $(p_\lambda)_{\lambda \in \Lambda}$ be a family of seminorms with $|\Lambda| \leq c$ which defines the topology of E. If Φ is the set of all non-empty finite subsets of Λ, we have $E' = \bigcup_{\substack{\phi \in \Phi \\ \epsilon > 0}} \{x : p_\lambda(x) \leq \epsilon,\ \lambda \in \phi\}^o$, which expresses E' as the union of at most c $\sigma(E', E)$-compact sets. Thus we can use Lemma 2.2.1 above to deduce that there is a subset A of B with cardinality at most c such that x is in the $\sigma(F, E')$-closure of A. Being a subset of a $\sigma(F, E')$-relatively compact set, A is $\sigma(F, E')$-relatively compact and so by Theorem 2.2.2 we have that A is essentially separable.

Remark

Let E and F be as we have in the above Corollary. Then Eberlein's theorem [43, Chapter VI, Theorem 4] gives a "natural" condition under which a subset A is $\sigma(F, E')$-relatively compact, namely if every sequence of points of A has a $\sigma(F, E')$-cluster point in F. We know also from a result given originally by R.C. James and in simplified form by J.D. Pryce [39, Theorem] that if E is real, a subset A is $\sigma(F, E')$-relatively compact if and only if each element of E' attains its supremum on the $\sigma(F, E')$-closure of A.

In [40, Theorem 4.2], J.D. Pryce shows that if E is a metrizable locally convex space with dual E', then for any topology of the dual pair (E, E') and any subset of E we have

(a) relative compactness, relative countable compactness and relative sequential compactness are equivalent,

(b) compactness, countable compactness and sequential compactness are equivalent

(c) each point in the closure of a relatively compact subset A is the limit of a sequence in A.

Applying the above Corollary, we obtain corresponding criteria for weak compactness and weak relative compactness in a separated locally convex space whose topology is defined by at most c seminorms.

Theorem 2.2.3

Let E be a separated locally convex space whose topology is defined by at most c seminorms. A subset B of E is $\sigma(E, E')$-relatively compact (respectively $\sigma(E, E')$-compact) if and only if each essentially separable subset of B is $\sigma(E, E')$-relatively compact (respectively $\sigma(E, E')$-relatively compact and has its

$\sigma(E, E')$-closure contained in B).

Proof

It is clear that the conditions are necessary. Suppose that either condition is satisfied. A countable set is trivially essentially separable and so B is $\sigma(E, E')$-relatively countably compact. Let F be the completion of E. Using Eberlein's theorem [28, §24, 2(1)] we have that B is $\sigma(F, E')$-relatively compact. It then follows from the Corollary to Theorem 2.2.2 that each $\sigma(F, E')$-point of closure x of B is already in E so that B is $\sigma(E, E')$-relatively compact under either condition. Under the bracketed condition, x is in B and so B is $\sigma(E, E')$-compact.

The following example shows that in Theorem 2.2.3 we cannot generally dispose of the condition that the topology be defined by at most c seminorms. It illustrates further that in general we cannot replace the hypothesis on the essentially separable sets by a weaker one on the separable subsets.

Example 2.2.1

Let M be a non-empty set and let

$$E = \{(\xi_\mu)_{\mu\in M} \in \mathbb{R}^M : |\{\mu : \xi_\mu \neq 0\}| \leq c\} ,$$

$$F = \{(\xi_\mu)_{\mu\in M} \in \mathbb{R}^M : |\{\mu : \xi_\mu \neq 0\}| \leq \aleph_0\} ,$$

$$G = \mathbb{R}^M ,$$

$$H = \mathbb{R}^{(M)} .$$

Then each of (E, H), (F, H) and (G, H) is a dual pair. Let $J = [0, 1]^M$ and put $A = J \cap E$ and $B = J \cap F$. We note that A (respectively B) is not $\sigma(E, H)$-relatively compact (respectively

$\sigma(F, H)$-relatively compact since both A and B are dense subsets of the $\sigma(G, H)$-compact set J.

We show first that if C is any subset of $\mathbb{R}^M$, and D is a dense subset of C, then supp C = supp D. Certainly, supp $D \subseteq$ supp C. Suppose there exists $\nu \in$ supp $C \backslash$ supp D and choose $(\xi_\mu)_{\mu\in M} \in C\backslash D$ such that $\xi_\nu \neq 0$. Since D is dense in C, there exists $(\eta_\mu)_{\mu\in M} \in D$ such that

$$|<(\xi_\mu)_{\mu\in M} - (\eta_\mu)_{\mu\in M}, (\delta_{\nu\mu})_{\mu\in M}>| < |\xi_\nu| ,$$

where

$$(\delta_{\nu\mu})_{\mu\in M} \in \mathbb{R}^{(M)} .$$

But $<(\xi_\mu)_{\mu\in M} - (\eta_\mu)_{\mu\in M}, (\delta_{\nu\mu})_{\mu\in M}> = \xi_\nu - \eta_\nu = \xi_\nu$, since $\nu \notin$ supp D. This contradiction shows that supp C = supp D.

Let X be any $\sigma(E, H)$-bounded essentially separable set. Since each element of E has support of cardinality at most c and since an essentially separable set has a weakly dense subset of cardinality at most c, we have that $|\text{supp } X| \leq c$. It then follows that there exists $M_o \in M$ such that $|M_o| \leq c$ and if $(\xi_\mu)_{\mu\in M} \in X$, $\xi_\nu = 0$ for all $\nu \in M\backslash M_o$. Hence $X \subseteq \pi_{\mu\in M} J_\mu$ where each J_μ is a compact interval in $\mathbb{R}$ and if $\mu \in M\backslash M_o$, $J_\mu = [0, 0] = \{0\}$. It therefore follows that $\pi_{\mu\in M} J_\mu \subseteq E$. Now by Tychonoff's theorem, $\pi_{\mu\in M} J_\mu$ is compact and consequently X is $\sigma(E, H)$-relatively compact. We have thus shown that each $\sigma(E, H)$-bounded essentially separable set is $\sigma(E, H)$-relatively compact.

If $|M| > c$, we have that A is a non-compact $\sigma(E, H)$-closed set in which each essentially separable subset is $\sigma(E, H)$-relatively compact. Consequently by Theorem 2.2.3 no topology of the dual pair (E, H) can be defined by at most c seminorms and it follows that the theorem may fail if the seminorm condition is removed. Furthermore, if we regard A as a subset of G, we see similarly that each essentially separable subset of A is $\sigma(G, H)$-relatively compact and has its $\sigma(G, H)$-closure contained in A. However A is not $\sigma(G, H)$-compact, being dense in J, although it is $\sigma(G, H)$-relatively compact.

Finally, let $|M| = c$. The topology $\sigma(F, H)$ is then defined by c seminorms. Again each countable subset of B is $\sigma(F, H)$-relatively compact and has its $\sigma(F, H)$-closure contained in B. However B is a $\sigma(F, H)$-closed essentially separable set which is not $\sigma(F, H)$-compact, being dense in J.

CHAPTER III

δ-BARRELLED SPACES

Mahowald's characterization [35, Theorem 2.2] of barrelled spaces as those which serve as domain spaces for a closed graph theorem in which the range space is an arbitrary Banach space can be regarded as the beginning of attempts to use the closed graph theorem to describe certain locally convex spaces. Kalton in [25], considered this kind of characterization when the range space is an arbitrary separable Banach space. Recently, V. Eberhardt [11, 14] described the locally convex spaces which can serve as domain spaces for the case when the range space is an arbitrary normed space. The corresponding domain spaces for the situation in which the range space is an arbitrary metrizable locally convex space have also been studied by V. Eberhardt and W. Roelcke in [15] (see also [11]). In the non-locally convex situation, Iyahen [21], [24] has characterized ultrabarrelled (respectively hyperbarrelled) spaces as those topological vector spaces (respectively semiconvex) spaces for which the closed graph theorem holds when the range space is any complete metric (respectively complete separated locally bounded) topological vector space.

This chapter is concerned primarily with the study of those locally convex spaces (δ-barrelled spaces) which can serve as domain spaces for a closed graph theorem when the range space is an arbitrary Banach space of dimension at most c. Many of the standard elementary Banach spaces, including all the separable ones, have dimension at most c. As we have already noted in Section 1.5, an infinite dimensional Banach space has dimension at least c. If we classify Banach spaces by dimension, we are therefore dealing in a natural sense, with the first class which contains infinite dimensional spaces. We give various characterizations of the δ-barrelled spaces (defined below) and we also establish some of their permanence properties (Corollaries, 1, 2 and 3 of Theorem 3.1.2). Conditions under which δ-barrelled spaces are barrelled are also investigated; in particular, we have an analogue of [52, Theorem 1].

3.1 Definitions and Basic Properties

Let E be a locally convex space. If A is a barrel in E, then $\bigcap_{\lambda>0} \lambda A$ is a closed vector subspace of E. A subset B of E is called a <u>δ-barrel</u> in E if it is a barrel such that the dimension of $E/\bigcap_{\lambda>0}\lambda B$ is at most c. We say that E is <u>δ-barrelled</u> if each δ-barrel in E is a neighbourhood of the origin in E.

It is obvious from the above definitions that every barrelled space is δ-barrelled. We shall show later by example that there are δ-barrelled spaces which are not barrelled even in the associated Mackey topology so that the class of barrelled spaces is a proper subclass of the class of δ-barrelled spaces.

The following theorem gives a characterization of separated δ-barrelled spaces in terms of essential separability which we introduced in the previous chapter.

<u>Theorem 3.1.1</u>

A separated locally convex space E is δ-barrelled if and only if each $\sigma(E', E)$-bounded essentially separable set is equicontinuous.

<u>Proof</u>

Suppose first that E is δ-barrelled. Let A be any non-empty $\sigma(E', E)$-bounded essentially separable subset of E' and let H be the linear span of A. Then A° is a barrel in E. Let C be the absolutely convex envelope of A. Then $A^o = C^o$ and $H = \bigcup_{\mu>0} \mu C$. Thus $H^o = \bigcap_{\mu>0} \frac{1}{\mu} C^o = \bigcap_{\lambda>0} \lambda A^o$. Hence if, as in Section 2.1, $\mathcal{N}(E, A)$ is the normed space constructed from A, then $\dim E/_{\bigcap_{\lambda>0} \lambda A^o} = \dim \mathcal{N}(E, A)$. Since A is $\sigma(E', E)$-bounded and essentially separable, by Theorem 2.1.3, we have that $\dim \mathcal{N}(E, A) \leq c$. Thus A^o is a δ-barrel in E and it therefore follows by the hypothesis that A^o is a neighbourhood of the origin in E. By [28, §21, 3(1)], we must have that A is equicontinuous.

Conversely, assume that the condition is satisfied. Let B be any δ-barrel in E. Then $(\bigcap_{\lambda>0} \lambda B)^\bullet$, the polar of $\bigcap_{\lambda>0} \lambda B$ in E^*, is $\sigma(E^*,E)$-separable, since the dimension of $E/_{\bigcap_{\lambda>0} \lambda B}$ is at most c and $(\bigcap_{\lambda>0} \lambda B)^\bullet$ is (isomorphic to) $(E/_{\bigcap_{\lambda>0} \lambda B})^*$. Let B^o be the polar of B in E'. Then since $B^o \subseteq (\bigcap_{\lambda>0} \lambda B)^\bullet$, we have that B^o is essentially separable. As B is absorbent, B^o is also $\sigma(E', E)$-bounded. Hence by the hypothesis, B^o is equicontinuous and so using [28, §21, 3(1)] again, $B = B^{oo}$ is a neighbourhood of the origin in E. This establishes the result.

<u>Remarks</u>

(i) It follows from the proof of Theorem 3.1.1 above that in a separated locally convex space E, the δ-barrels are precisely the polars of the $\sigma(E', E)$-bounded essentially separable sets.

(ii) In a separated locally convex space E, every $\sigma(E', E)$-bounded sequence is a $\sigma(E', E)$-bounded essentially separable set. It then follows from the above characterization of δ-barrelledness that every separated δ-barrelled space is σ-barrelled (Section 1.3). As we shall see (Example 3.1.1(b)), there are σ-barrelled spaces which are not δ-barrelled.

(iii) We note also that every δ-barrelled space of dimension at most c is barrelled, for, in this case since $\dim E/_{\bigcap_{\lambda>o}\lambda B} \leq \dim E \leq c$, for each barrel B, it follows that every barrel in E is a δ-barrel.

(iv) We observe further that, from the above theorem, it follows that a separated δ-barrelled space is δ-barrelled under any finer topology of the same dual pair.

The next theorem provides our principal reason for introducing δ-barrelled spaces. It corresponds to Mahowald's well-known characterization of barrelled spaces [35, Theorem 2.2] which has already been mentioned. First we establish the following lemma which we need for the theorem.

Lemma 3.1.1

Let E, F be locally convex spaces and let $t : E \to F$ be a linear mapping. If B is a δ-barrel in F, then $\overline{t^{-1}(B)}$ is a δ-barrel in E.

Proof

Certainly $\overline{t^{-1}(B)}$ is a barrel in E. Let $M = \bigcap_{\lambda>o}\lambda(\overline{t^{-1}(B)})$ and $N = \bigcap_{\lambda>o}\lambda B$. Suppose that $x_1 + M, x_2 + M, \ldots, x_n + M$ are linearly independent in $E/_M$. We must have that $t(x_1) + N$, $t(x_2) + N, \ldots, t(x_n) + N$ are linearly independent in $F/_N$, for if they are not, we can choose scalars α_r $(r = 1, \ldots, n)$, which are not all zero such that $\sum_{r=1}^{n} \alpha_r(t(x_r) + N) = 0$ in $F/_N$. It then follows that $t(\sum_{r=1}^{n} \alpha_r x_r) \in N$. We must then have that

$$\sum_{r=1}^{n} \alpha_r x_r \in t^{-1}(N) = t^{-1}(\bigcap_{\lambda>o}\lambda B) = \bigcap_{\lambda>o}\lambda\, t^{-1}(B) \subseteq \bigcap_{\lambda>o}\lambda(\overline{t^{-1}(B)}) = M,$$

which contradicts the fact that $x_1 + M, \ldots, x_n + M$ are linearly independent in $E/_M$. It therefore follows that $\dim E/_M \leq \dim F/_N \leq c$. Thus $\overline{t^{-1}(B)}$ is a δ-barrel in E .

A similar argument establishes

Lemma 3.1.2

Let E, F be locally convex spaces and let $t : E \to F$ be a linear surjection. If B is a δ-barrel in E , then $\overline{t(B)}$ is a δ-barrel in F .

Theorem 3.1.2

A locally convex space E is δ-barrelled if and only if whenever F is a Banach space of dimension at most c and $t : E \to F$ is a linear mapping with a closed graph, t is necessarily continuous.

Proof

Suppose first that E is δ-barrelled. Let $t : E \to F$ be a linear mapping with a closed graph of E into a Banach space F of dimension at most c . Let B be the closed unit ball of F . By Lemma 3.1.1 and Remark (iii) following Theorem 3.1.1, $\overline{t^{-1}(B)}$ is a δ-barrel in E and so by the hypothesis it is a neighbourhood of the origin in E . It therefore follows by [26, 11.1] that t is continuous.

To establish the converse, suppose that E satisfies the closed graph condition and let B be a δ-barrel in E . Let $N = \bigcap_{\lambda>0} \lambda B$ and $q : E \to E/_N$ be the quotient map. The Minkowski functional of $q(B)$ is a norm on $E/_N$. Let G be the completion of $E/_N$ under the topology defined by this norm. Since B is a δ-barrel, the dimension of $E/_N$ is at most c , from which it follows by Lemma 2.1.4 that the dimension of G is at most c . As is shown in the proof of Lemma 3.1 of [21], the linear mapping $k : E \to G$ defined by $k(x) = q(x)$ has a closed graph. By the hypothesis therefore, k is continuous. Thus $k^{-1}(k(B))$ is a neighbourhood of the origin in E. Since $k^{-1}(k(B)) = B + N \subseteq B + B = 2B$, it follows that B is a neighbourhood of the origin in E and so E is δ-barrelled as required.

The following corollaries give some basic permanence properties of δ-barrelled spaces.

Corollary 1

Any inductive limit of δ-barrelled spaces is δ-barrelled.

Proof

Let E be the inductive limit of the δ-barrelled spaces $E_\lambda (\lambda \epsilon \Lambda)$ by the linear mappings $t_\lambda (\lambda \epsilon \Lambda)$. Let $t : E \to F$ be a linear mapping with a closed graph of E into a Banach space F of dimension at most c. Then $t \circ t_\lambda : E_\lambda \to F$ is a linear mapping with a closed graph for each $\lambda \in \Lambda$, for if (x_α) is a net in E_λ such that (x_α) converges to x in E_λ, and $(t(t_\lambda(x_\alpha)))$ converges to y in F, then by continuity, we have that $(t_\lambda(x_\alpha))$ converges to $t_\lambda(x)$ and by the closedness of the graph of t, we deduce that $y = t(t_\lambda(x))$. Thus by the theorem, $t \circ t_\lambda$ is continuous for each $\lambda \in \Lambda$. Using [43, Chapter 5, Proposition 5], we get that t is continuous and again by the theorem, E must be δ-barrelled.

We note that in the separated case, Corollary 1 may also be deduced from the theorem using [23, Theorem 2.1].

Corollary 2

Any product of δ-barrelled spaces is δ-barrelled.

Proof

In the separated case this follows from the theorem and [23, Theorem 2.2]. In the non-separated case, we can use Eberhardt's generalization of Iyahen's result [14, Lemma 1.4] with the method of [14, Theorem 1.3].

Corollary 3

The completion $\hat{E}$ of a δ-barrelled space E is also δ-barrelled.

Proof

Suppose that $t : \hat{E} \to F$ is a linear mapping with a closed graph of $\hat{E}$ into a Banach space F with dimension at most c. The restriction $t|_E$, of t to E has a closed graph and so is continuous by the theorem. Let s be the unique extension of $t|_E$ to a continuous linear mapping of $\hat{E}$ into F [27, Chapter I, §8.5, Theorem 1]. Since s is continuous and the graph of t is closed, it follows that the linear mapping $s - t : \hat{E} \to F$ has a closed graph. If we define X by $X = \{x \in \hat{E} : s(x) = t(x)\}$ then X contains E. Besides, if (x_α) is a net in X converging to x, then since $(s - t)(x_\alpha) = 0$, we have that $(s - t)(x_\alpha)$ converges to 0. From the fact that $s - t$ has a closed graph, it follows that $(s - t)(x) = 0$. Thus $s(x) = t(x)$ and so $x \in X$. Hence X is a closed vector subspace of $\hat{E}$ which contains the dense subspace E. Consequently X must be the whole of $\hat{E}$ and so t is continuous. The result now follows from the theorem.

Remark

We can deduce from the closed graph theorem, in the usual way, that any linear mapping with a closed graph of a Banach space of dimension at most c onto a δ-barrelled space is open. To do this we only need to observe that any quotient of such a Banach space by a closed vector subspace (in this case the null-space of the mapping) is also a Banach space of dimension at most c.

The next result will generally allow us to deduce properties of δ-barrelled spaces (not necessarily separated) from the separated case.

Theorem 3.1.3

A locally convex space E is δ-barrelled if and only if $E/_N$ (with the quotient topology) is δ-barrelled, where N is the closure of $\{0\}$ in E.

Proof

If E is δ-barrelled, then it follows from Corollary 1 of Theorem 3.1.2 that $E/_N$ must be δ-barrelled.

Suppose conversely that $E/_N$ is δ-barrelled. Let B be a δ-barrel in E and let $\mathcal{U}$ be a base of absolutely convex neighbourhoods of the origin in E. Then $N = \bigcap_{U \in \mathcal{U}} U$ and if $q : E \to E/_N$ is the quotient map, by Lemma 3.1.2, we have that $\overline{q(B)}$ is a δ-barrel in $E/_N$. Since $E/_N$ is δ-barrelled it follows that $\overline{q(B)}$ is a neighbourhood of zero in $E/_N$. Therefore, there exists an absolutely convex neighbourhood V of zero in E such that $q(V) \subseteq \overline{q(B)}$. We then have that

$$V \subseteq q^{-1}(\overline{q(B)}) = q^{-1}\Big(\bigcap_{U \in \mathcal{U}} q(B + U)\Big) = \bigcap_{U \in \mathcal{U}} q^{-1}(q(B + U))$$

$$\subseteq \bigcap_{U \in \mathcal{U}} (B + 2U) = B, \quad \text{since } B \text{ is closed.}$$

Thus it follows that B is a neighbourhood of zero in E and so E is δ-barrelled.

Note

In the proof of Theorem 3.1.3 above, $q(B)$ is in fact closed in $E/_N$. This is effectively established in the course of the proof.

It is well known that if there is a continuous almost open linear mapping of the barrelled space E into a locally convex space F, than F is necessarily barrelled. The corresponding assertion for δ-barrelled spaces holds as the next result shows.

Theorem 3.1.4

Let E and F be locally convex spaces such that E is δ-barrelled and there is a continuous almost open linear mapping of E into F. Then F is δ-barrelled.

Proof

Let $f : E \to F$ be a continuous almost open linear mapping of E into F and let $g : F \to H$ be a linear mapping with a closed graph of F into a Banach space H of dimension at most c. Then clearly, $g \circ f : E \to H$ is a linear mapping with a closed graph of E into H. By Theorem 3.1.2, we have that $g \circ f$ is continuous. If B is the closed unit ball of H, we then have that $f^{-1}(g^{-1}(B))$ is a neighbourhood of the origin in E. Since f is almost open, it follows that $\overline{f(f^{-1}(g^{-1}(B)))} = \overline{g^{-1}(B)}$ is a neighbourhood of the origin in F. Thus by [26, 11.1], we have that g is continuous. Theorem 3.1.2 now shows that F is δ-barrelled.

Examples 3.1.1

(a) Let $E = \mathbb{R}^{(M)}$, where $|M| > c$,

$$E' = \{(\xi_\mu)_{\mu\in M} \in \mathbb{R}^M : |\{\mu : \xi_\mu \neq 0\}| \le c\} \quad .$$

Then (E, E') is a dual pair and $E^* = \mathbb{R}^M$. As we have seen in Example 2.2.1., every $\sigma(E', E)$-bounded essentially separable set is $\sigma(E', E)$-relatively compact. It therefore follows that $(E, \tau(E, E'))$ is a δ-barrelled space. However $(E, \tau(E, E'))$ is not barrelled, for if it were, $[0, 1]^M \cap E'$ which is $\sigma(E', E)$-bounded would be equicontinuous. This is false since $[0, 1]^M \cap E'$ is dense in the compact set $[0, 1]^M$ which is not contained in E' and so the $\sigma(E', E)$-closure of $[0, 1]^M \cap E'$ cannot be $\sigma(E', E)$-compact.

(b) Let $E = \mathbb{R}^{(M)}$, where $|M| = c$,

$$E' = \{(\xi_\mu)_{\mu\in M} \in \mathbb{R}^M : |\{\mu : \xi_\mu \neq 0\}| \leq \aleph_0\} .$$

Again (E, E') is a dual pair with $E^* = \mathbb{R}^M$. Let $\mathcal{A}$ be the family of all subsets of E' of the form

$$\{(\xi_\mu)_{\mu\in M} : |\xi_\mu| \leq \alpha_\mu \ (\mu\in\Lambda) , \ \xi_\mu = 0 \ \text{otherwise}\} ,$$

where Λ is an at most countable subset of M and for each $\mu \in \Lambda$ α_μ is a non-negative real number.

Since each element of $\mathcal{A}$ is a $\sigma(E', E)$-compact absolutely convex set and since the union of the elements of $\mathcal{A}$ is E', it follows from the Mackey-Arens Theorem that $\mathcal{J}$, the topology on E of uniform convergence on the elements of $\mathcal{A}$, is a topology of the dual pair (E, E'). We have: (i) $(E, \mathcal{J})$ is countably barrelled.

<u>Proof</u>

Let $B_n (n \in \mathbb{N})$ be equicontinuous sets and suppose that $\bigcup_{n=1}^{\infty} B_n$ is $\sigma(E', E)$-bounded. Since the set of polars of elements of $\mathcal{A}$ forms a base of neighbourhoods of the origin for $\mathcal{J}$, it follows that each B_n has an at most countable support so that $\bigcup_{n=1}^{\infty} B_n$ has at most countable support. Since $\bigcup_{n=1}^{\infty} B_n$ is $\sigma(E', E)$-bounded, for each $\mu_0 \in M$, we have that $\sup \{|\xi_{\mu_0}| : (\xi_\mu)_{\mu\in M} \in \bigcup_{n=1}^{\infty} B_n\}$ exists. It therefore follows that $\bigcup_{n=1}^{\infty} B_n$ is contained in an element of $\mathcal{A}$ and so is $\mathcal{J}$-equicontinuous. Hence $(E, \mathcal{J})$ is countably barrelled as required.

(ii) E cannot have a δ-barrelled topology for the dual pair (E, E') .

<u>Proof</u>

Since $|M| = c$, by Theorem 1.4.1, we have that $E^* = \mathbb{R}^M$ is $\sigma(E^*, E)$-separable. Thus $[0, 1]^M \cap E'$ is a $\sigma(E', E)$-bounded essentially separable set. But $[0, 1]^M \cap E'$ is not equicontinuous since $[0, 1]^M \cap E'$ cannot be $\sigma(E', E)$-relatively compact being dense in $[0, 1]^M$. It therefore follows that no topology of the dual pair (E, E') is δ-barrelled.

As we have pointed out earlier a countably barrelled space is also σ-barrelled. Hence the space $(E, \mathcal{J})$ in (b) above is a σ-barrelled space which is not δ-barrelled.

(c) Let Ω be the first uncountable ordinal and consider the space $[0, \Omega)$ of all ordinals less than Ω . We assert first that $C([0, \Omega))$ has dimension c . To see this we observe that by [17, 5.12(c)], every f in $C([0, \Omega))$ is constant on a tail, that is on a set of the form $\{x \in [0, \Omega) : x \geq \sigma\}$ for some ordinal σ less than Ω . Also there are at most c tails and each tail has at most $\aleph_0$ predecessors. It therefore follows that there are at most c ways of making a continuous real-valued function with a given tail. Hence the cardinality and consequently the dimension of $C([0, \Omega))$ cannot exceed c . Now, $C([0, \Omega))$ and $C(\beta[0, \Omega)) = C[0, \Omega])$ are algebraically isomorphic so that the dimension of $C([0, \Omega))$ and the dimension of $C([0, \Omega])$ are equal. But $C_c([0, \Omega])$ is an infinite dimensional Banach space. It therefore follows that the dimension of $C([0, \Omega))$ is at least c and so $C([0, \Omega))$ has dimension c .

From the last paragraph of §7 of [36], we know that $C_c([0, \Omega))$ is σ-barrelled but not barrelled. We note also that $\{\delta_x : x \in [0, \Omega)\}$, where δ_x is the point measure with mass 1 at x, is a bounded (essentially separable) subset of $C([0, \Omega))'$ which is not compact in $[C([0, \Omega))']_\sigma$. This follows from the fact that $(\delta_x)_{x\in[0, \Omega)}$ is a Cauchy net in $[C([0, \Omega))']_\sigma$ which converges weakly to the linear functional x' defined by $x'(f) = \lim_{x\to\Omega} f(x)$ and $x' \notin C([0, \Omega))'$ since it has no support in $[0, \Omega)$. It now follows that $C([0, \Omega))$ cannot be δ-barrelled for any topology of the dual pair $(C([0, \Omega)), C([0, \Omega))')$.

It is interesting to note however that for a sufficiently large ordinal γ, we can have that $C_c([0, \gamma))$ is δ-barrelled. For example, let Γ be any cardinal such that

(i) $\Gamma > c$

(ii) there exists $\Gamma' < \Gamma$ such that if $\Delta < \Gamma$, then $\Delta \le \Gamma'$.

We may for example take Γ to be the least cardinal which is strictly greater than c.

Now let γ be the least ordinal of cardinality Γ. Let $X \subseteq [0, \gamma)$ and suppose that $|X| \le c$. We show that X has an upper bound in $[0, \gamma)$. If $\alpha \in X$, than $\alpha + 1 \in [0, \gamma)$ since otherwise $\alpha + 1 = \gamma$ and so we would have that α and γ have the same cardinality which is false. Hence we have

$$X \subseteq \bigcup_{\alpha\in X} [0, \alpha + 1) \subseteq [0, \gamma) \quad \text{and}$$

$$\left|\bigcup_{\alpha\in X} [0, \alpha + 1)\right| \le \sum_{\alpha\in X} |[0, \alpha + 1)| \le \sum_{\alpha\in X} \Gamma' \le c \,.\, \Gamma' = \Gamma' < \Gamma \,.$$

Thus we can choose $\beta \in [0, \gamma)\setminus(\bigcup_{\alpha\in X}[0, \alpha + 1))$ and this β is an upper bound for X in $[0, \gamma)$.

Let A be any bounded essentially separable set in $[C([0, \gamma))']_\sigma$. Then by the Corollary to Theorem 2.1.5, there exists a dense subset

$\{\mu_d : d \in D\}$ of A with $|D| \leq c$. For each $d \in D$, supp μ_d is compact and so there exists $\alpha_d \in [0, \gamma)$ such that supp $\mu_d \subseteq [0, \alpha_d]$. Let β be an upper bound for $\{\alpha_d : d \in D\}$ in $[0, \gamma)$. Then we have that supp A = supp D = $\mathrm{cl}(\bigcup_{d \in D} \text{supp } \mu_d) \subseteq [0, \beta] \subseteq [0, \gamma)$, where $\mathrm{cl}(X)$ denotes the closure of X . It follows that A is equicontinuous [57, Lemma 4] and consequently $C_c([0,\gamma))$ is δ-barrelled.

(d) Consider the dual pair (ℓ_∞, ℓ_1) . The space $(\ell_\infty, \tau(\ell_\infty, \ell_1)$ is not δ-barrelled. This is because the closed unit ball of ℓ_1 is a $\sigma(\ell_1, \ell_\infty)$-separable bounded set which is not $\sigma(\ell_1, \ell_\infty)$-relatively compact.

We recall that, according to Kalton [25], $\mathcal{B}(\zeta_B)$ is the class of all separated locally convex spaces E with the property that whenever $t : E \to F$ is a linear mapping with a closed graph of E into an arbitrary separable Banach space F , t is continuous. As mentioned earlier, every infinite dimensional separable Banach space has dimension c . The converse is not true; for example, ℓ_∞ has dimension c but it is not separable in its usual norm. Our separated δ-barrelled spaces therefore form a subclass of the class $\mathcal{B}(\zeta_B)$. In [25], N.J. Kalton showed that $(\ell_\infty, \tau(\ell_\infty, \ell_1))$ is in $\mathcal{B}(\zeta_B)$. As we have seen above, $(\ell_\infty, \tau(\ell_\infty, \ell_1))$ is not δ-barrelled. Hence the separated δ-barrelled spaces constitute a proper subclass of Kalton's $\mathcal{B}(\zeta_B)$.

A. Wilansky [58] has announced the following variant of Mahowald's characterization of barrelled spaces. A locally convex space E is barrelled if (and only if) every linear mapping with a closed graph of E into $C_c(X)$ is continuous, where X is an arbitrary compact Hausdorff space. We establish the corresponding result for δ-barrelled spaces. First we give the following lemma which is needed for the result.

Lemma 3.1.3

Let E be a separated locally convex space and let A be a $\sigma(E', E)$-bounded subset of E'. If B is the $\sigma(E^*, E)$-closed absolutely convex envelope of A, then the mapping $t : E \to C_c(B)$ defined by $(t(x))(b) = \langle x, b\rangle$ for all $b \in B$ is linear and has a closed graph.

Proof

That t is linear is obvious. Let (x_λ) be a net in E converging to some $x \in E$ and let $(t(x_\lambda))$ converge to f in $C_c(B)$. We have to show that $f = t(x)$, that is $f(b) = \langle x, b\rangle$ for all b in B. Now $(t(x_\lambda))$ converges to f in $C_c(B)$ implies that $(t(x_\lambda))(b)$ converges to $f(b)$ for all b in B. If C is the absolutely convex envelope of A, then C is $\sigma(E^*, E)$-dense in B. Also since (x_λ) converges to x in E, it follows that $(\langle x_\lambda, x'\rangle)$ converges to $\langle x, x'\rangle$ for all $x' \in E'$ and so $f(b) = (t(x))(b)$ for all $b \in C$. Thus $(f - t(x))(b) = 0$ for all $b \in C$. Since C is $\sigma(E^*, E)$-dense in B and $f - t(x)$ is continuous, we have that $(f - t(x))(b) = 0$ for all $b \in B$ and so $f = t(x)$.

Theorem 3.1.5

Let E be a locally convex space. Then E is δ-barrelled if and only if whenever X is a compact Hausdorff space such that the dimension of $C_c(X)$ is at most c and $t : E \to C_c(X)$ is a linear mapping with a closed graph, then t is continuous.

Proof

The necessity of the condition follows immediately from Theorem 3.1.2 since $C_c(X)$ is a Banach space of dimension at most c.

Suppose conversely that the condition is satisfied. We consider first the case when E is separated. Let A be a non-empty $\sigma(E', E)$-bounded essentially separable set and let B be the $\sigma(E^*, E)$-closed absolutely convex envelope of A. Then B is compact and Hausdorff under the topology induced by $\sigma(E^*, E)$. We show that the Banach space $C_c(B)$ has dimension at most c. By Theorem 2.1.3, the normed space $\mathcal{N}(E, A)$ constructed from A has dimension at most c. It therefore follows that the vector subspace $\{x|_B : x \in E\}$ of $C_c(B)$ has dimension at most c and hence its cardinality is also at most c. By the Stone-Weierstrass theorem [10, Chapter XIII, Theorem 3.3 and page 283], we have that the subalgebra generated by $\{x|_B : x \in E\} \cup \{1\}$ in the real case or $\{x|_B : x \in E\} \cup \{\overline{x}|_B : x \in E\} \cup \{1\}$ in the complex case is dense in $C_c(B)$ in its norm topology. Similar reasoning as we have in the proof of Theorem 2.2.1, now gives that the dimension of $C_c(B)$ is at most c.

Now consider the mapping $t : E \to C_c(B)$ defined by $(t(x))(b) = \langle x, b\rangle$ for all x in E. By Lemma 3.1.3, t is a linear mapping with a closed graph. Hence by the hypothesis, t is continuous and so its transpose $t' : C(B)' \to E'$ maps equicontinuous sets into equicontinuous sets. If δ_b is the point measure with mass 1 at b, then since $\{\delta_b : b \in B\}$ is a subset of the closed unit ball of $C(B)$, we have that $t'(\{\delta_b : b \in B\})$ is equicontinuous. But for all $x \in E$, and all $b \in B$, we have

$$\langle x, t'(\delta_b)\rangle = \langle t(x), \delta_b\rangle = (t(x))(b) = \langle x, b\rangle .$$

Thus $t'(\{\delta_b : b \in B\}) = B$ and so $B \subseteq E'$. Consequently, A is equicontinuous and therefore E is δ-barrelled.

If E is not separated, by considering the closure N of $\{0\}$ and the quotient space $E/_N$, it will follow from the first part and Theorem 3.1.3 that E is δ-barrelled. This completes the proof.

Of recent, the inheritance of properties of certain locally convex spaces by vector subspaces of countable (i.e. at most countable) codimension has been of interest to some Mathematicians. It is now known for example that a vector subspace of countable codimension of a barrelled, σ-barrelled, or countably barrelled space is again of the same type ([46, Main theorem], [54, Theorem 3], [56, Theorem 6], [31, §4 Theorem]). The analogous result for δ-barrelled spaces is also true as the following theorem shows.

Theorem 3.1.6

Let E be a δ-barrelled space and let X be a subspace of E of countable codimension. Then X is δ-barrelled in the induced topology.

Proof

We consider first the separated case. Since each $\sigma(E', E)$-bounded sequence forms an equicontinuous set, E' is $\sigma(E', E)$-sequentially complete. If G is the closure in E of X, then G also has countable codimension in E and so we deduce from [31, §3, Proposition] that G' is $\sigma(G', G)$- sequentially complete. Let A be a $\sigma(X', X)$-bounded essentially separable set and let B be the subset of G' obtained by extending by continuity each element of A. By [31, §2, Lemma], B is $\sigma(G', G)$-bounded.

If $G \neq E$ we can choose an at most countable family $\{x_n\}$ of linearly independent elements of $E \setminus G$ which spans a supplement of G in E. In this case we extend the elements of B to the whole of E by putting $\langle x_n, x'\rangle = 0$ for each n and each $x' \in B$. Let C be the set of all these extensions. By the Lemma of [46, §2], we have that $C \subseteq E'$ and it is clear that C is $\sigma(E', E)$-bounded. If $G = E$, put $C = B$.

We show next that C is essentially separable. Since the result is trivial if $X = \{0\}$, we may assume that $X \neq \{0\}$. In the usual way, X^* is topologically isomorphic to K^M where $|M|$ is the dimension of X. Since X has countable codimension in E, this topological isomorphism extends to a topological isomorphism of E^* onto $K^{M \cup N}$, for some at most countable index set N such that $M \cap N = \emptyset$. Let $M_o \subseteq M$ and $\{x_\mu'\}$ be as we have in Theorem 2.1.4 for the image of A in K^M, which is essentially separable by Lemma 2.1.2. Now $|M_o \cup N| \leq c$ and if for each $\mu \in (M \cup N) \setminus (M_o \cup N) = M \setminus M_o$ we define an element of $K^{(M \cup N)}$ by the same formula as defines x_μ' except that $e_{\alpha(r, \mu)}$ now lies in $K^{(M \cup N)}$, we can apply Theorem 2.1.4 to the image of C in $K^{M \cup N}$. It follows from Lemma 2.1.2 again that C is essentially separable for the dual pair (E', E). By the hypothesis therefore, C is equicontinuous and so C^o is a neighbourhood of the origin in E. Hence $C^o \cap X$ is a neighbourhood of the origin in the topology induced from E. But $C^o \cap X = A^o$ and consequently A is equicontinuous. Thus X is δ-barrelled.

If E is not separated, let N be the closure of $\{0\}$ in E and let $L = N \cap X$. Now $X/_L$ is topologically isomorphic to the subspace $\{x + N : x \in X\}$ of $E/_N$ which clearly has an at most countable codimension in $E/_N$. The result now follows from the separated case and Theorem 3.1.3.

Later, we shall give a slightly different proof of Theorem 3.1.6 in a more general setting.

3.2 The δ-topology of a δ-barrelled Space

In this section, we discuss the δ-topology of a δ-barrelled space and use it to obtain some illustrative examples of δ-barrelled spaces. We begin with two lemmas which are needed for some of the results in this section.

The first lemma follows easily from the fact that a countable union of countable sets is again countable and the fact that the rational (respectively complex rational) numbers are dense in $\mathbb{R}$ (respectively $\mathbb{C}$).

Lemma 3.2.1

Let X_n $(n \in \mathbb{N})$ be separable subsets of a topological vector space E and let Y be the closed absolutely convex envelope of $\bigcup_{n=1}^{\infty} X_n$. Then Y is separable.

Lemma 3.2.2

Let $\mathcal{B}$ be an at most countable set of δ-barrels in a locally convex space E . If $B_o = \cap\{B : B \in \mathcal{B}\}$ is absorbent, then B_o is a δ-barrel.

Proof

Any intersection of closed absolutely convex sets is closed and absolutely convex. It follows therefore that B_o is a barrel in E . It only remains to show that the dimension of $E/_{\bigcap\limits_{\lambda>o}\lambda B_o}$ is at most c . We note first that

$$\bigcap_{\lambda>o}\lambda B_o = \bigcap_{\lambda>o}\lambda(\cap\{B : B \in \mathcal{B}\}) = \cap\{\bigcap_{\lambda>o}\lambda B : B \in \mathcal{B}\} .$$

Now for each $B \in \mathcal{B}$, we have that $E/_{\bigcap\limits_{\lambda>o}\lambda B}$ has dimension at most c and so by Theorem 1.5.1, $(E/_{\bigcap\limits_{\lambda>o}\lambda B})^*$ is separable under

$\sigma((E/\bigcap_{\lambda>0}\lambda B)^*, E/\bigcap_{\lambda>0}\lambda B)$. Hence for each $B \in \mathcal{B}$, we have that $(\bigcap_{\lambda>0}\lambda B)^\bullet$ is separable under $\sigma(E^*, E)$. By [26, 16.3 (vii)], $(\bigcap_{\lambda>0}\lambda B_0)^\bullet = (\bigcap\{\bigcap_{\lambda>0}\lambda B : B \in \mathcal{B}\})^\bullet$ is the $\sigma(E^*, E)$-closed absolutely convex envelope of $\cup\{(\bigcap_{\lambda>0}\lambda B)^\bullet : B \in \mathcal{B}\}$. Since $\mathcal{B}$ is at most countable and since $(\bigcap_{\lambda>0}\lambda B)^\bullet$ is $\sigma(E^*, E)$-separable for each $B \in \mathcal{B}$, it follows from Lemma 3.2.1 that $(\bigcap_{\lambda>0}\lambda B_0)^\bullet$ is $\sigma(E^*, E)$-separable. Consequently, by Theorem 1.5.1 again, we have that $E/\bigcap_{\lambda>0}\lambda B_0$ has dimension at most c and so B_0 is a δ-barrel in E .

For the definition of the δ-topology of a δ-barrelled space, we need the following result.

<u>Theorem 3.2.1</u>

Let (E, ξ) be a δ-barrelled space. Then the set of all δ-barrels in (E, ξ) forms a base of neighbourhoods of the origin for a coarser locally convex topology $\delta(\xi)$ under which E is both δ-barrelled and countably barrelled.

<u>Proof</u>

Let $\mathcal{U}$ be the set of all δ-barrels in (E, ξ) . We show first that $\mathcal{U}$ satisfies the conditions C1 - C3 of [43, Chapter I, Theorem 2]. Since each $U \in \mathcal{U}$ is absolutely convex and absorbent, $\mathcal{U}$ satisfies C3. Any scalar multiple of a barrel is a barrel and for any barrel B and any non-zero scalar α , we have $\bigcap_{\lambda>0}\lambda(\alpha B) = \bigcap_{\lambda>0}\lambda B$. Hence if $U \in \mathcal{U}$ and α is a non-zero scalar, then $\alpha U \in \mathcal{U}$, which is the condition C2. It is clear from Lemma 3.2.2 that if $U, V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$ and this gives C1. By [43, Chapter I, Theorem 2], there is a locally convex topology, $\delta(\xi)$ say with $\mathcal{U}$ as a base of neighbourhoods of the origin in E .

It is clear, from the fact that (E,ξ) is δ-barrelled that $\delta(\xi)$ is coarser than ξ. If B is a δ-barrel in $(E, \delta(\xi))$, then since ξ is finer than $\delta(\xi)$, we have that B is a δ-barrel in (E, ξ). It therefore follows that B is a $\delta(\xi)$-neighbourhood of the origin in E. Thus E is δ-barrelled under $\delta(\xi)$.

We show next that E is countably barrelled under $\delta(\xi)$. Suppose that $B = \bigcap_{n=1}^{\infty} U_n$ is absorbent, where for each n, U_n is a closed absolutely convex $\delta(\xi)$-neighbourhood of the origin in E. We note that since each U_n contains a δ-barrel B_n, we have that U_n is itself a δ-barrel for each n, as in this case by Lemma 1.5.1 $\dim E/\bigcap_{\lambda>o} \lambda U_n \leq \dim E/\bigcap_{\lambda>o} \lambda B_n \leq c$. It follows from Lemma 3.2.2 that B is a δ-barrel in $(E, \delta(\xi))$ and consequently it is a $\delta(\xi)$-neighbourhood of the origin. Hence $(E, \delta(\xi))$ is countably barrelled.(Here we use Theorem 1 of [20] to extend the concept of countable barrelledness to the non-separated case).

<u>Definition</u>

Let (E, ξ) be a δ-barrelled space. The topology $\delta(\xi)$ with the set of all δ-barrels as a base of neighbourhoods of the origin (Theorem 3.2.1) will be called the <u>δ-topology</u> of (E, ξ).

When (E, ξ) is a separated δ-barrelled space, we have the following useful characterization of the δ-topology $\delta(\xi)$ of (E, ξ).

<u>Theorem 3.2.2</u>

Let (E, ξ) be a separated δ-barrelled space. Then the δ-topology $\delta(\xi)$ of (E, ξ) is the topology of uniform convergence on the family $\mathcal{A}$ of all $\sigma(E', E)$-bounded essentially separable sets. The set of polars of elements of $\mathcal{A}$ forms a base of neighbourhoods of the origin for $\delta(\xi)$.

Consequently $\delta(\xi)$ is separated and it is a topology of the dual pair (E, E') .

Proof

Let $\delta(E, E')$ be the topology of uniform convergence on the elements of $\mathcal{A}$. Since $\mathcal{A}$ is closed under the formation of finite unions and scalar multiples [Lemma 2.1.1], the polars of the elements of $\mathcal{A}$ form a base of neighbourhoods for $\delta(E, E')$ [43, Chapter III, §2]. It follows from the fact that $\mathcal{A}$ contains all finite subsets of E' that $\delta(E, E')$ is finer than $\sigma(E, E')$ and consequently $\delta(E, E')$ is separated. Since E is δ-barrelled in the original topology, each element of $\mathcal{A}$ is equicontinuous in the original topology and therefore by the Mackey-Arens theorem [43, Chapter III, Theorem 7], the $\sigma(E', E)$-closed absolutely convex envelope of each element of $\mathcal{A}$ is $\sigma(E', E)$-compact and so $\delta(E, E')$ is a topology of the dual pair (E, E') .

That $\delta(E, E') = \delta(\xi)$ follows immediately from Remark (i) immediately after Theorem 3.1.1.

In the separated case, the δ-topology of a δ-barrelled space (E, ξ) with dual E' will be denoted by $\delta(E, E')$ as in the proof of Theorem 3.2.2 above.

Let E be a separated barrelled space with dual E' . Then E is δ-barrelled and its topology is $\tau(E, E')$. The δ-topology $\delta(E, E')$ can be strictly coarser than $\tau(E, E')$. We give two examples.

<u>Example 3.2.1</u>

(i) Consider the dual pair $(\ell_\infty', \ell_\infty'')$ discussed in Example 2.1.2. Under its norm topology, ℓ_∞' is barrelled and therefore δ-barrelled. The δ-topology $\delta(\ell_\infty', \ell_\infty'')$ is however strictly coarser than $\tau(\ell_\infty', \ell_\infty'')$, for if $\tau(\ell_\infty', \ell_\infty'') = \delta(\ell_\infty', \ell_\infty'')$, then the closed unit ball B of ℓ_∞'' would be $\delta(\ell_\infty', \ell_\infty'')$-equicontinuous. Theorem 3.2.2 would then tell us that B is essentially separable for the dual pair $(\ell_\infty'', \ell_\infty')$. We have seen in Example 2.1.2 that this is not the case. Since the closed unit ball A of ℓ_∞ is $\sigma(\ell_\infty'', \ell_\infty')$-dense in B, it follows that A is not $\delta(\ell_\infty', \ell_\infty'')$-equicontinuous.

In [53], Valdivia calls a separated locally convex space E a γ-barrelled space if each $\sigma(E', E)$-bounded set of cardinality at most γ is equicontinuous. Since $|A| = c$, the above example shows that a separated δ-barrelled space need not be c-barrelled. On the other hand a c-barrelled space is necessarily δ-barrelled, for by the Corollary to Theorem 2.1.5, every set which is essentially separable for the dual pair (E', E) has a $\sigma(E', E)$-dense subset of cardinality at most c. Consequently, if X is a $\sigma(E', E)$-bounded essentially separable set, it contains a $\sigma(E', E)$-bounded dense subset Y of cardinality at most c. Hence if E is c-barrelled, then $Y^\circ = \bar{Y}^\circ$ is a neighbourhood of the origin in E and since $\bar{Y}^\circ = X^\circ$, we have that X is equicontinuous. Thus from Theorem 3.1.1, it follows that E is δ-barrelled.

(ii) Let Γ be a set with cardinality strictly greater than c. The Hilbert space $\ell_2(\Gamma)$ has dimension strictly greater than c. Under its δ-topology $\delta(\ell_2(\Gamma), \ell_2(\Gamma))$, the space $\ell_2(\Gamma)$ is δ-barrelled but not barrelled, since the closed unit ball X of $\ell_2(\Gamma)$ is $\sigma(\ell_2(\Gamma), \ell_2(\Gamma))$-bounded but not $\delta(\ell_2(\Gamma), \ell_2(\Gamma))$-equicontinuous since

$$\mathcal{N}(\ell_2(\Gamma), X) = \ell_2(\Gamma) \quad .$$

We note also that $\ell_2(\Gamma)$ is c-barrelled under $\delta(\ell_2(\Gamma)\ \ \ell_2(\Gamma))$. To see this, let A be a $\sigma(\ell_2(\Gamma), \ell_2(\Gamma))$-bounded set with cardinality at most c . Applying Theorem 2.2.2 for the norm topology on $\ell_2(\Gamma)$ shows that A is essentially separable and consequently equicontinuous under $\delta(\ell_2(\Gamma), \ell_2(\Gamma))$.

In Theorem 3.2.1, we showed that a δ-barrelled space (E, ξ) is both δ-barrelled and countably barrelled under its δ-topology $\delta(\xi)$. We now give an example of a δ-barrelled space which is not countably barrelled. In [48, Proposition 4.4], J. Schmets describes a general method of constructing σ-barrelled spaces which are not countably barrelled. We adapt his technique in our example, although our approach is rather different.

<u>Example 3.2.2</u>

Let M be a non-empty set and let

$$E = \mathbb{R}^{(M)}$$

$$E' = \{(\xi_\mu)_{\mu\in M} \in \mathbb{R}^M : |\{\mu : \xi_\mu \neq 0\}| \leq c\} .$$

For each non-empty subset B of M define $S(B)$ by

$$S(B) = \{(\xi_\mu)_{\mu\in M} \in \mathbb{R}^M : \xi_\mu = 0 \quad \text{if} \quad \mu \notin B , \sum_{\mu\in M} |\xi_\mu| \leq 1\} .$$

Now, $\sum_{\mu\in M} |\xi_\mu| \leq 1$ implies that we only have $\xi_\mu \neq 0$ for at most countably many μ and so $S(B) \subseteq E'$. It is clear that $S(B)$ is $\sigma(E', E)$-bounded and absolutely convex. We show that it is also closed. Let $(\xi_\mu)_{\mu\in M}$ be in the closure of $S(B)$ in $\mathbb{R}^M$. Let ϕ be any non-empty finite subset of M and suppose that ϕ has n elements. Given $\varepsilon > 0$, there exists $(\eta_\mu)_{\mu\in M}$ in $S(B)$ such that

$$|\xi_\mu - \eta_\mu| \leq \frac{\varepsilon}{n} , \text{ for all } \mu \in \phi .$$

Thus we have

$$\sum_{\mu\in\phi} |\xi_\mu| \le \sum_{\mu\in\phi} |\xi_\mu - \eta_\mu| + \sum_{\mu\in\phi} |\eta_\mu| \le \epsilon + 1 \ ,$$

from which it follows that $\sum_{\mu\in\phi} |\xi_\mu| \le 1$, since $\epsilon > 0$ is arbitrary. Hence $\sum_{\mu\in M} |\xi_\mu| \le 1$ since ϕ is an arbitrary non-empty finite subset of M . It is easily seen that $\xi_\mu = 0$ if $\mu \notin B$. Consequently, $(\xi_\mu)_{\mu\in M} \in S(B)$ and so $S(B)$ is closed. Thus $S(B)$ is compact in $\mathbb{R}^M$ and since it is contained in E' , it is $\sigma(E', E)$-compact.

Now take $M = \mathcal{P}(\mathbb{R})$, the power set of $\mathbb{R}$. Then $|M| = 2^c$ so that $E' \neq \mathbb{R}^M$, and as we have seen in Example 3.1.1(a), it follows that $(E, \tau(E, E'))$ is δ-barrelled but not barrelled. Let $\mathcal{B}$ be the collection of all $\sigma(E', E)$-bounded essentially separable sets together with the sets $S(\mathcal{P}(C))$ where C is a compact subset of $\mathbb{R}$. Let ξ be the topology of uniform convergence on the sets in $\mathcal{B}$. From the above observations and the Mackey-Arens theorem, we have that ξ is a topology of the dual pair (E, E') and that a base of neighbourhoods of the origin for ξ is given by all sets of the form $D^o \cap \epsilon\, S(\mathcal{P}(C))^o$, where D is a non-empty $\sigma(E', E)$-bounded essentially separable set, $\epsilon > 0$ and C is a compact subset of $\mathbb{R}$ (see for example [43, Chapter III, §2]). It is clear from Remark (iv) after Theorem 3.1.1 and Theorem 3.2.2 that ξ is a δ-barrelled topology.

Let $A = \bigcup_{n=1}^{\infty} S(\mathcal{P}([-n, n]))$. Then being contained in $S(\mathcal{P}(\mathbb{R}))$, A is a $\sigma(E', E)$-bounded set which is the union of a sequence of ξ-equicontinuous sets. As was noted in Example 2.2.1, if X is essentially separable for the dual pair (E', E) , then $|\text{supp } X| \le c$. Using this and the fact that $|\mathcal{P}([-n, n]) \setminus \mathcal{P}(C)| = 2^c$ for all sufficiently large n , given any set V of the form

$D^{\circ} \cap \epsilon S(\mathcal{P}(C))^{\circ}$ above, we may choose

$\nu \in (\overset{\infty}{\underset{n=1}{U}} \mathcal{P}([-n, n]))\setminus\{(\text{supp } D) \cup \mathcal{P}(C)\}$. Then $(2\,\delta_{\mu\nu})_{\mu\epsilon M}$ is in V so that $(\delta_{\mu\nu})_{\mu\epsilon M} \notin V^{\circ}$. Since $(\delta_{\mu\nu})_{\mu\epsilon M} \in A$, this shows that A is not ξ-equicontinuous and consequently (E, ξ) is not countably barrelled. Thus (E, ξ) is a δ-barrelled space which is not countably barrelled.

<u>Note</u>

We have seen that δ-barrelledness is preserved in forming inductive limits, products and completions [Corollaries 1, 2 and 3 of Theorem 3.1.2]. Also δ-barrelledness is inherited by subspaces of countable codimension [Theorem 3.1.6]. It is natural to enquire how the δ-topology behaves under these operations.

It follows from Lemma 3.1.2 that a quotient topology obtained from a δ-topology is of the same type. However the direct sum of δ-topologies may fail to be a δ-topology e.g. $\mathbb{R}^{(M)}$ with $|M| > c$ (c.f. Lemma 2.1.6). Lemmas 3.1.1 and 3.2.2 show that a product of δ-topologies is a δ-topology (see [43, Chapter V, §5]). Corollary 1 of Theorem 2.1.7 shows that the completion of a space with a δ-topology also has a δ-topology. Clearly the intersection of a δ-barrel and a vector subspace is a δ-barrel in the vector subspace. Consequently the topology induced by a δ-topology on a vector subspace of countable codimension is a δ-topology.

3.3 <u>Conditions under which a δ-barrelled space is barrelled</u>

This section is concerned with conditions under which a δ-barrelled space is barrelled or has a finer barrelled topology of the same dual pair. As we have already observed in Remark (ii) after Theorem 3.1.1, a separated δ-barrelled space is σ-barrelled. Corollary 4(a) of [9] also tells us that a separable σ-barrelled space is barrelled. We deduce from this and the analogue of Theorem 3.1.3 for barrelled spaces that a separable δ-barrelled space is barrelled.

Our first result gives conditions under which a separated δ-barrelled space E is barrelled under its Mackey topology $\tau(E, E')$.

<u>Theorem 3.3.1</u>

Let E be a separated δ-barrelled space with completion $\hat{E}$. Suppose that there is a family $(X_\lambda)_{\lambda\in\Lambda}$ of subsets of E such that

(i) $|\Lambda| \le c$,

(ii) $\cup\{X_\lambda : \lambda\in\Lambda\}$ is total in E under $\beta(E, E')$,

(iii) for each $\lambda\in\Lambda$, X_λ is $\sigma(\hat{E}, E')$-relatively compact.

Then E is barrelled under $\tau(E, E')$.

<u>Proof</u>

Let Y_λ be the $\sigma(\hat{E}, E')$-closed absolutely convex envelope of $X_\lambda (\lambda\in\Lambda)$. By Krein's theorem [28, §24, 5(4)] and the condition (iii), each Y_λ is $\sigma(\hat{E}, E')$-compact. Denote by G the subspace of $\hat{E}$ spanned by $\cup\{Y_\lambda : \lambda\in\Lambda\}$ and let H be the subspace of E spanned by $\cup\{X_\lambda : \lambda\in\Lambda\}$.

Let B be a $\sigma(E', E)$-bounded closed set and let A be a subset of B which is essentially separable for the dual pair (E', G). Now A is equicontinuous for the topology $\xi = \beta(E, E')|_H$ and essentially separable for the dual pair (E', H) (Corollary to Theorem 2.1.1). Since E is contained in the ξ-completion of H, it follows from Corollary 2 of Theorem 2.1.7 that A is essentially separable for the dual pair (E', E). Since E is δ-barrelled, A is equicontinuous. If C is the $\sigma(E', E)$-closure of A, then $C \subseteq B$ and by [43, Chapter VI, Corollary to Theorem 2], we have that C is also $\sigma(E', \hat{E})$-compact. It follows also that C is $\sigma(E', G)$-

compact since $\sigma(E', \hat{E})$ is finer than $\sigma(E', G)$. Since $\cup\{Y_\lambda : \lambda\in\Lambda\}$ spans G , the set of scalar multiples of finite intersections of polars of $Y_\lambda (\lambda\in\Lambda)$ forms a base of neighbourhoods for a separated topology η on E' and since each Y_λ is $\sigma(G, E')$-compact and absolutely convex, this topology η is a topology of the dual pair (E', G) . Besides since $|\Lambda| \leq c$, the set of all scalar multiples of finite intersections of polars of $Y_\lambda (\lambda\in\Lambda)$ has cardinality at most c and so the topology η is defined by at most c seminorms. It therefore follows by Theorem 2.2.3 that B is $\sigma(E', G)$-compact. This implies that B is $\sigma(E', H)$-compact. Since B is ξ-equicontinuous and since E is contained in the ξ-completion of H we have that B is $\sigma(E', E)$-compact. It now follows that E is barrelled under $\tau(E, E')$.

Corollary

Let E be a separated δ-barrelled space which contains a $\beta(E, E')$-dense subset of cardinality at most c . Then E is barrelled.

Proof

Let $\{x_\lambda : \lambda\in\Lambda\}$ be a $\beta(E, E')$-dense subset of E with $|\Lambda| \leq c$. Taking $X_\lambda = \{x_\lambda\}$ $(\lambda\in\Lambda)$ in the theorem we deduce that $(E, \tau(E, E'))$ is δ-barrelled. However by Corollary 2 of Theorem 2.1.7, each $\sigma(E', E)$-bounded set is essentially separable for the dual pair (E', E). Thus the initial topology of E is $\tau(E, E')$.

Remark

In Theorem 3.3.1 above, the initial δ-barrelled topology on E need not be the Mackey topology $\tau(E, E')$. Take for example $\ell_2(\Gamma)$ of Example 3.2.1 (ii) with $|\Gamma| > c$. Then $(\ell_2(\Gamma), \delta(\ell_2(\Gamma), \ell_2(\Gamma)))$ is δ-barrelled but not barrelled. However the conditions of Theorem 3.3.1 are satisfied by taking the unit ball of $\ell_2(\Gamma)$ as the single X_λ .

We recall the following definitions. Let (E, ξ) be a topological vector space and let M be an index set. We denote by Φ, the set of all non-empty finite subsets of M . The family $(z_\mu)_{\mu \in M}$ of elements of E is said to be unconditionally Cauchy if for each ξ-neighbourhood U of the origin, there exists $\phi_1 \in \Phi$ such that $\sum_{\mu \in \phi} z_\mu \in U$ for all $\phi \in \Phi$ with $\phi_1 \cap \phi = \emptyset$. We say that $(z_\mu)_{\mu \in M}$ is unconditionally convergent to z if for each such neighbourhood U , there exists $\phi_2 \in \Phi$ such that $z - \sum_{\mu \in \phi} z_\mu \in U$ for all $\phi \in \Phi$ with $\phi_2 \subseteq \phi$. We then write this as $z = \sum_{\mu \in M} z_\mu$.

Our next theorem which involves the concept of unconditional convergence is an analogue of Theorem 1 of [52]. First we establish a lemma which is probably well known and the proof of which we include for completeness.

Lemma 3.3.1

Let E be a σ-barrelled space. If $\sum_{\lambda \in \Lambda} x_\lambda$ converges unconditionally in E , it also converges unconditionally under $\beta(E, E')$ to the same sum.

Proof

It is enough to show that $\sum_{\lambda \in \Lambda} x_\lambda$ is unconditionally Cauchy under $\beta(E, E')$, for then the result will follow from [28, §18, 4(4)] since $\beta(E, E')$ has a base of neighbourhoods of the origin which are $\sigma(E, E')$-closed.

Suppose on the contrary that $\sum_{\lambda \in \Lambda} x_\lambda$ is not unconditionally Cauchy under $\beta(E, E')$. Let Φ be the set of all non-empty finite subsets of Λ . Then there is a $\sigma(E', E)$-bounded set B such that for each

$\phi \in \Phi$, there exist $\phi' \in \Phi$ with $\phi' \cap \phi = \emptyset$ and $x'_{\phi'} \in B$ such that

$$|<\sum_{\lambda\in\phi'} x_\lambda , x'_{\phi'}>| > 1 \qquad (*) .$$

We now take $\phi = \phi_o \in \Phi$ and choose $\phi' = \phi_1$ and $x'_{\phi'} = x_1'$ such that (*) holds. Then put $\phi = \phi_o \cup \phi_1$ and $\phi' = \phi_2$ and $x'_{\phi'} = x'_2$ such that (*) holds. Putting $\phi = \phi_o \cup \phi_1 \cup \phi_2$ and continuing in this way, we obtain a sequence (ϕ_n) in Φ and a sequence (x_n') in B such that $\phi_{n+1} \cap (\bigcup_{r=1}^{n} \phi_r) = \emptyset$ and

$$|<\sum_{\lambda\in\phi_n} x_\lambda , x'_n>| > 1 , \quad (n \in \mathbb{N}) \qquad (**) .$$

Then since E is σ-barrelled, $\{x_n' : n \in \mathbb{N}\}$ is equicontinuous. Hence $\sum_{\lambda\in\Lambda} x_\lambda$ being unconditionally Cauchy in E , there exists $\phi \in \Phi$ such that if $\phi' \in \Phi$ and $\phi' \cap \phi = \emptyset$, we have

$$|<\sum_{\lambda\in\phi} x_\lambda , x_n'>| \le 1 , \quad (n \in \mathbb{N}) .$$

This contradicts (**) since ϕ is finite and so $\phi \cap \phi_n = \emptyset$ for all sufficiently large n . Thus $\sum_{\lambda\in\Lambda} x_\lambda$ is unconditionally Cauchy under $\beta(E, E')$.

Theorem 3.3.2

Let E be a separated δ-barrelled space and suppose that there is a family $(x_\mu)_{\mu\in M}$ of elements of E such that

(a) for each x in E , there exists scalars α_μ $(\mu\in M)$ such that $\sum_{\mu\in M} \alpha_\mu x_\mu$ is unconditionally convergent to x ,

(b) there is a family $(z_\lambda)_{\lambda\in\Lambda}$ of elements of E such that $z_\lambda = \sum_{\mu\in M} \alpha_\mu^{(\lambda)} x_\mu$ $(\lambda\in\Lambda)$, $|\Lambda| \le c$ and for each $\mu\in M$ at least one $\alpha_\mu^{(\lambda)} \neq 0$.

Then each $\sigma(E', E)$-bounded set is essentially separable and consequently E is barrelled.

<u>Proof</u>

Let A be a non-empty $\sigma(E', E)$-bounded set. For each x in E let $\bar{x}$ denote its equivalence class in the normed space $\mathcal{N}(E, A)$ constructed from A (Section 2.1). If $\sum_{\mu\in M} \alpha_\mu x_\mu$ converges unconditionally to x, then since a δ-barrelled space is also σ-barrelled, it follows from Lemma 3.3.1 that $\sum_{\mu\in M} \alpha_\mu x_\mu$ converges to x unconditionally under $\beta(E, E')$. Thus $\sum_{\mu\in M} \alpha_\mu \bar{x}_\mu$ converges unconditionally to $\bar{x}$ in $\mathcal{N}(E, A)$. Since $\mathcal{N}(E, A)$ is a normed space, we use [7, §5, Corollary to Proposition 1] to conclude that the set $\{\mu \in M : \alpha_\mu \bar{x}_\mu \neq 0\}$ is at most countable. Consequently, $\bigcup_{\lambda\in\Lambda}\{\mu \in M : \alpha_\mu^{(\lambda)} \bar{x}_\mu \neq 0\}$ has cardinality at most c. By the condition (b) this set is $\{\mu \in M : \bar{x}_\mu \neq 0\}$. Since the linear span of $\{x_\mu : \mu \in M\}$ is $\beta(E, E')$-dense in E, $\{\bar{x}_\mu : \mu \in M\}$ is total in $\mathcal{N}(E, A)$. By Lemma 2.1.4, we must have that the dimension of $\mathcal{N}(E, A)$ is at most c. Consequently by Theorem 2.1.3, we have that A is essentially separable for the dual pair (E', E) and the result is established.

<u>Remark</u>

The space E of Theorem 3.3.2 need not have a dense subset of cardinality at most c. For example let $E = \mathbb{R}^M$, where $|M| > 2^c$. Then by Theorem 1.4.2, we know that E has no dense subset of cardinality at most c. However, we may apply Theorem 3.3.2 to $\mathbb{R}^M$ with $x_\mu = (\delta_{\mu\gamma})_{\gamma\in M}$ and a single z_λ, namely $\sum_{\mu\in M} x_\mu$.

3.4 The Associated δ-barrelled Topology

Given a locally convex space E, the associated barrelled topology on E is defined and studied by Y. Kōmura in [29]. Various aspects of the associated barrelled topology of a locally convex space have also been discussed by N. Adasch [1], A. Robert [45] and V. Eberhardt [12]. In this section we define the associated δ-barrelled topology of a locally convex space. We establish some results which are analogues of corresponding results for the associated barrelled topology. Some of our techniques are similar to those used in [12].

Definition 3.4.1

Let (E, ξ) be a locally convex space. We define, by transfinite induction, a topology t_γ on E for each ordinal γ as follows:

(i) $t_o = \xi$;

(ii) if $\gamma > 0$ is not a limit ordinal, let t_γ be the upper bound topology of $t_{\gamma-1}$ and the locally convex topology on E having as base of neighbourhoods of zero all the $t_{\gamma-1}$ - δ-barrels;

(iii) if γ is a limit ordinal, let t_γ be the upper bound topology of $\{t_\sigma : \sigma < \gamma\}$.

It should be noted that all the t_γ cannot be distinct (otherwise $|\mathcal{P}(E)|$ would be arbitrarily large). Also if $t_\gamma = t_{\gamma+1}$, then $t_\sigma = t_\gamma$ for all $\sigma \geq \gamma$.

Now let $\gamma_o = \inf\{\gamma : t_\sigma = t_\gamma ,$ for all $\sigma \geq \gamma\}$ and let $\delta_\xi = t_{\gamma_o}$. Then $t_\sigma = \delta_\xi$ for all $\sigma \geq \gamma$ and δ_ξ is a δ-barrelled topology on E finer than ξ . Let η be any δ-barrelled topology on E finer than ξ . It is clear that if t_γ is coarser than η

for some ordinal γ, then the same is true for $t_{\gamma+1}$. Thus we must have that δ_ξ is coarser than η. Hence δ_ξ is the coarsest δ-barrelled topology on E which is finer than the initial topology ξ. The topology δ_ξ is called the associated δ-barrelled topology for (E, ξ).

When (E, ξ) is a separated locally convex space with dual E', we adapt Adasch's hull idea in [1] to give a dual characterization of the associated δ-barrelled topology in the following way.

Definition 3.4.2

For each vector subspace H of E', let H^δ be the intersection of all vector subspaces G of E^* such that

(i) $H \subseteq G$,

(ii) the $\sigma(E^*, E)$-closure of each $\sigma(E^*, E)$-bounded subset of G which is essentially separable for the dual pair (E^*, E) is contained in G.

Consider the dual pair $(E, (E')^\delta)$. If A is a $\sigma((E')^\delta, E)$-bounded essentially separable set, then by the definition of $(E')^\delta$, the $\sigma(E^*, E)$-closure of A is contained in $(E')^\delta$ and so A is $\sigma((E')^\delta, E)$-relatively compact. Thus E is δ-barrelled under $\tau(E, (E')^\delta)$. If s is the upper bound topology of ξ and $\delta(E, (E')^\delta)$, then s is the coarsest δ-barrelled topology of the dual pair $(E, (E')^\delta)$ which is finer than ξ. If η is a δ-barrelled topology on E which is finer than ξ, then the dual of (E, η) is one of the subspaces G of E^* in the intersection defining $(E')^\delta$. Thus η is finer than s and consequently $s = \delta_\xi$.

Remark

We note that it follows from the above dual characterization of the associated δ-barrelled topology that if E is a separated locally convex space and ξ , η are topologies of the dual pair (E, E'), then δ_ξ and δ_η are topologies of the same dual pair.

The following is an analogue of (1.4) of [12].

Theorem 3.4.1

Let t be a continuous linear mapping of a δ-barrelled space (E, ξ) into a locally convex space (F, η) . Then $t : (E,\xi) \to (F, \delta_\eta)$ is also continuous.

Proof

Let ζ be the locally convex topology on F having a base of neighbourhoods of the origin consisting of all absolutely convex subsets A of F such that $t^{-1}(A)$ is a ξ-neighbourhood of the origin in E . Since t is continuous under ξ and η , it follows that ζ is finer than η . Let B be a δ-barrel in (F, ζ) . Since t is also continuous under ξ and ζ , we have from Lemma 3.1.1 that $t^{-1}(B)$ is a δ-barrel in E and so is an ξ-neighbourhood of the origin. Thus B is a ζ-neighbourhood of the origin in F and hence (F, ζ) is δ-barrelled. It now follows that ζ is finer than δ_η and consequently that t is continuous under ξ and δ_η.

Corollary

Let (E, ξ) and (F, η) be separated locally convex spaces. If the linear mapping $t: (E, \xi) \to (F, \eta)$ is weakly continuous and (E, ξ) is δ-barrelled, then $t : (E, \xi) \to (F, \delta_\eta)$ is also weakly continuous.

Proof

Let E' be the dual of E under ξ. From [43, Chapter III, Proposition 14], it follows that $t : (E, \tau(E, E')) \to (F, \eta)$ is continuous. Since E is δ-barrelled for all topologies of the dual pair (E, E') finer than $\delta(E, E')$ it follows from the theorem that $t : (E, \tau(E, E')) \to (F, \delta_\eta)$ is continuous. Since ξ and $\tau(E, E')$ are topologies of the same dual pair, the desired weak continuity of t follows from [43, Chapter II, Proposition 13].

We end this section with a lemma which is an analogue of (1.2) of [12]. The proof is similar to that of (1.2) of [12] and so will be omitted.

Lemma 3.4.1

Let (E, ξ) be a separated locally convex space. Let F (respectively G) be the completion of E under ξ (respectively under δ_ξ). If $\hat{i} : G \to F$ is the extension by continuity of the identity mapping of (E, δ_ξ) onto (E, ξ) then $\hat{i}$ is one-to-one.

CHAPTER IV

δ-SPACES AND INFRA-δ-SPACES

Consider the linear mapping $t : E \to F$ with a closed graph of a locally convex space E into a locally convex space F. We have seen in Chapter III that if t is continuous whenever F is a Banach space of dimension at most c, then this is equivalent to the δ-barrelledness of E. It is natural to ask for those separated locally convex spaces which can serve as range spaces for a closed graph theorem in which the domain space is an arbitrary δ-barrelled space. This Chapter is concerned mainly with describing such locally convex spaces when in addition the domain space is assumed to have its Mackey topology.

Many Mathematicians have considered similar problems for the case when the domain space is an arbitrary barrelled space. Such consideration led to the concept of B_r-completeness discussed by V. Pták in [42]. T. Husain in [18] and [19] introduced his class of $B(\mathcal{J})$ -spaces and gave various characterizations of these spaces. The classes of t-polar and weakly t-polar spaces were introduced by Persson in [38]. In [29, §3], Kōmura treated the closed graph theorem and defined his concept of minimal topology which was adapted by N. Adasch to define his s- and infra-s-spaces which are discussed in [1].

We adapt Adasch's hull idea in [1] to describe those separated locally convex spaces (infra-δ-spaces) which can serve as range spaces for our closed graph theorem. The corresponding domain spaces (δ-spaces) for the open mapping theorem in which the range is a δ-barrelled space with its Mackey topology are also discussed. Examples will be given to show that δ-spaces and infra-δ-spaces form proper subclasses of the classes of s- and infra-s-spaces respectively. We shall see also that infra-δ-spaces need not be weakly t-polar.

Definition 4.1.1

Let F be a separated locally convex space. If H is a vector subspace of F', we define H^{δ} as we have in Section 3.4 of Chapter 3. We say that

(a) F is a δ-space if the $\sigma(F', F)$-closure of each vector subspace H of F' coincides with $F' \cap H^{\delta}$,

(b) F is an infra-δ-space if for each $\sigma(F', F)$-dense vector subspace H of F' we have that $F' \subseteq H^{\delta}$.

Let F be a separated locally convex space. For any vector subspace H of F', we must have that $F' \cap H^{\delta}$ is always contained in the $\sigma(F', F)$-closure of H. To see this, let G be the $\sigma(F^*, F)$-closure of H. Since the $\sigma(F^*, F)$-closure of any $\sigma(F^*, F)$-bounded subset of G is contained in G, it follows from the definition of H^{δ} that G is one of the subspaces in the intersection determining H^{δ}. Hence $H^{\delta} \subseteq G$ and consequently $F' \cap H^{\delta} \subseteq F' \cap G$ which is the $\sigma(F', F)$-closure of H.

Before discussing the properties of the δ-spaces and the infra-δ-spaces, we look at some examples.

Example 4.1.1

(a) Every Fréchet space F with dimension at most c is a δ-space, for if H is any vector subspace of F' and U is any neighbourhood of the origin in F, we have that $U^{\circ} \cap H^{\delta} \cap F'$ is a $\sigma(F^*, F)$-bounded subset of H^{δ}. It is also essentially separable since F has dimension at most c. By the construction of H^{δ}, the $\sigma(F^*, F)$-closure A of $U^{\circ} \cap H^{\delta} \cap F'$ must be contained in H^{δ}. Since U° is

$\sigma(F', F)$-compact, it is $\sigma(F^*, F)$-closed. Thus A is also contained in U^o. We now have that $A \subseteq U^o \cap H^\delta = U^o \cap H^\delta \cap F'$ so that $U^o \cap H^\delta \cap F'$ is $\sigma(F', F)$-closed. Since F is B-complete, and since U was an arbitrary neighbourhood of the origin in F, it follows that $H^\delta \cap F'$ is $\sigma(F', F)$-closed. Since $H \subseteq H^\delta \cap F'$ we have $\overline{H} \subseteq H^\delta \cap F'$. But as noted above, $H^\delta \cap F'$ is contained in the $\sigma(F', F)$-closure $\overline{H}$ of H. Thus $H^\delta \cap F' = \overline{H}$.

It should be noted however that an arbitrary complete locally convex space of dimension at most c need not be a δ-space. For example, if E is a Banach space of dimension c, then E under the topology $\tau(E, E^*)$ is a complete locally convex space. Consider E' as a subspace of E^*. Then $(E')^\delta = E'$ but the $\sigma(E^*, E)$-closure of E' is the whole of $E^* \neq E'$. Thus $(E, \tau(E, E^*))$ is not a δ-space, nor an infra-δ-space (c.f. [43, Chapter VI, Supplement (1)]).

(b) More generally, arguing in the same way, we see that if F is a B-complete space (respectively a t-polar space) such that the equicontinuous subsets of F' (respectively $\sigma(F', F)$-bounded sets) are essentially separable, then F is a δ-space. Similar observations hold with B-complete, t-polar, δ-space replaced by B_r-complete, weakly t-polar, infra-δ-space respectively.

We adapt the method of [13, Example 1] to obtain our next example. First we state the following lemma the proof of which is effectively contained in [37, Theorem 2.1].

<u>Lemma 4.1.1</u>

Let $Y = \pi_{\alpha\in A} Y_\alpha$ be a product of first countable topological vector spaces and let X be the vector subspace

$$\{(\xi_\alpha)_{\alpha\in A} \in Y : |\{\alpha : \xi_\alpha \neq 0\}| \leq \aleph_0\} \text{ of } Y .$$

Then whenever x is in the closure in X of a subset S of X, we have that S contains a sequence which converges to x.

(c) Let $E = \mathbb{R}^{(M)}$, where M is a non-empty set,

$$E' = \{(\xi_\mu)_{\mu \in M} \in \mathbb{R}^M : |\{\mu : \xi_\mu \neq 0\}| \leq \aleph_0\} .$$

Then E is a δ-space for any topology of the dual pair (E, E').

<u>Proof</u>

Let H be any vector subspace of E' and let (x'_n) be a sequence in $H^\delta \cap E'$ which converges to x' under $\sigma(E', E)$. Since $\{x'_n : n \in \mathbb{N}\}$ is (essentially) separable, its $\sigma(E^*, E)$-closure must be contained in each G considered in constructing H^δ. Thus $x' \in H^\delta$ and so $H^\delta \cap E'$ is $\sigma(E', E)$-sequentially closed. Since $\mathbb{R}^M$ is a product of first countable topological vector spaces, it follows from Lemma 4.1.1 above that $H^\delta \cap E'$ is $\sigma(E', E)$-closed. We now deduce as in (a) that the $\sigma(E', E)$-closure of H coincides with $H^\delta \cap E'$. Thus E is a δ-space.

Now let $\mathcal{J}$ be the topology on E of uniform convergence on the family $\mathcal{A}$ of sets of the form $\prod_{\mu \in M} [-a_\mu, a_\mu]$ where at most countably many $a_\mu \neq 0$. If H is a vector subspace of E' such that $H \cap A$ is $\sigma(E', E)$-closed for any set $A \in \mathcal{A}$, then arguing as in [13, (2)], we have that H is $\sigma(E', E)$-closed. Thus $(E, \mathcal{J})$ is B-complete. More generally, by [43, Chapter VI, Proposition 8], we have that E is B-complete under any topology of the dual pair (E, E') which is finer than $\mathcal{J}$.

NOTE

Since the property of being weakly t-polar is a property of the dual pair and since B_r-complete spaces are necessarily weakly t-polar, the promised example of an infra-δ-space which is not weakly t-polar (Example 4.1.3) will provide an example of an infra-δ-space which is not B_r-complete for any topology of its dual pair.

The following is an analogue of [1, §5, (4)].

Theorem 4.1.1

Let F be an (infra-) δ-space and let G be a $\sigma(F', F)$-dense vector subspace of F'. Then F is an (infra-) δ-space for any topology of the dual pair (F, G).

Proof

Let H be any vector subspace of G. We note that H^{δ} is the same for the dual pairs (F, F') and (F, G) since it depends only on H, F and F^*. Let $\overline{H}$ (respectively $\tilde{H}$) denote the $\sigma(F', F)$-closure (respectively $\sigma(G, F)$-closure) of H. Then $\tilde{H} = \overline{H} \cap G$ and if F is a δ-space, we have $\overline{H} = H^{\delta} \cap F'$. Hence $\tilde{H} = (H^{\delta} \cap F') \cap G = H^{\delta} \cap G$. Thus F is a δ-space for the dual pair (F, G).

Since G is $\sigma(F', F)$-dense in F', it follows that any $\sigma(G, F)$-dense vector subspace H of G is also a $\sigma(F', F)$-dense vector subspace of F'. Thus if F is an infra-δ-space, we have that $H^{\delta} \cap F' = F'$ and consequently $H^{\delta} \cap G = G$. Hence F is an infra-δ-space for the dual pair (F, G).

As illustrated by Exercise 11 of [47, Chapter IV], the quotient of a complete space by a closed vector subspace need not be complete. However the quotient of a B-complete space by a closed vector subspace is also B-complete [47, Chapter IV, §8, Corollary to Theorem 8.3]. From [1, §5, (5)], we know that a quotient by a closed vector subspace also inherits the property of being an s-space. The next result shows that this property is also shared by δ-spaces.

Theorem 4.1.2

Let F be a δ-space and let M be a closed vector subspace of F. Then F/M is a δ-space under its quotient topology.

Proof

We note first that $(F/M)'$ is (isomorphic to) M° and that $(F/M)^*$ is (isomorphic to) $M^{\bullet}$. Let H be a vector subspace of M° and denote by H^{δ_1}, H^{δ_2} the hulls of H constructed for the dual pairs $(F/M, M^{\circ})$ and (F, F') respectively. To establish the result, we have to show that the $\sigma(M^{\circ}, F/M)$-closure of H coincides with $H^{\delta_1} \cap M^{\circ}$. Since $\sigma(F', F)$ and $\sigma(M^{\circ}, F/M)$ coincide on M°, it suffices to show that $H^{\delta_1} = H^{\delta_2}$, since the closure of H under $\sigma(M^{\circ}, F/M)$ coincides with its closure under $\sigma(F', F)$ and its closure under $\sigma(F', F)$ is $H^{\delta_2} \cap F' = H^{\delta_2} \cap M^{\circ}$.

Now let G be any vector subspace of $M^{\bullet}$ which contains H and suppose that the $\sigma(M^{\bullet}, F/M)$-closure of each $\sigma(M^{\bullet}, F/M)$-bounded subset of G which is essentially separable for the dual pair $(M^{\bullet}, F/M)$ is contained in G. Let A be a $\sigma(F^*, F)$-bounded subset of G which is essentially separable for the dual pair (F^*, F). It follows from Lemma 2.1.3 that A is essentially separable for the dual pair $(M^{\bullet}, F/M)$ so that the $\sigma(M^{\bullet}, F/M)$-closure B of A is contained in G. Now B is $\sigma(M^{\bullet}, F/M)$-compact and since $\sigma(M^{\bullet}, F/M)$ and $\sigma(F^*, F)$ coincide on $M^{\bullet}$, it follows that B is $\sigma(F^*, F)$-closed. Hence every vector subspace G of $M^{\bullet}$ considered in constructing H^{δ_1} is already one of the vector subspaces of F^* considered in the construction of H^{δ_2}. Thus $H^{\delta_2} \subseteq H^{\delta_1}$.

On the other hand, let G be any vector subspace of F^* which contains H and suppose that the $\sigma(F^*, F)$-closure of each $\sigma(F^*, F)$-bounded subset of G which is essentially separable for the dual pair (F^*, F) is contained in G. Since H^{**} is $\sigma(F^*, F)$-complete, $G \cap H^{**}$ has the same property. Reasoning as before, we have that $F^{\delta_1} \subseteq F^{\delta_2}$ so that $F^{\delta_1} = F^{\delta_2}$ and the proof is complete.

The next two results are analogues of [illegible] of [illegible] and [illegible] of [illegible]. The proofs make use of some of the techniques of the corresponding results in [illegible]. Our results characterise Mackey spaces which are δ-barrelled and they give a precise description of the range spaces in our closed graph theorem in which a δ-barrelled Mackey space is the domain space.

Theorem [illegible]

(i) A linear mapping with a closed graph of a δ-barrelled Mackey space into an infra-δ-space is continuous.

(ii) Let F be a separated locally convex space with the property that any linear mapping with a closed graph of a δ-barrelled Mackey space into F is continuous. Then F is an infra-δ-space.

(iii) Let E be a Mackey space with the property that any linear mapping with a closed graph of E into an infra-δ-space is continuous. Then E is δ-barrelled.

Proof

(i) Let $t : E \to F$ be a linear mapping with a closed graph of a δ-barrelled Mackey space E into an infra-δ-space F. Let $t^* : F^* \to E^*$ be the algebraic transpose of t. Then t^* is continuous under $\sigma(F^*, F)$ and $\sigma(E^*, E)$. If $H = t^{*-1}(E') \cap F'$, then trivially $H \subseteq t^{*-1}(E')$ and also if B is any $\sigma(F^*, F)$-bounded essentially separable subset of $t^{*-1}(E')$, by Lemma [illegible], we have that $t^*(B)$ is a $\sigma(E^*, E)$-bounded essentially separable

On the other hand, let G be any vector subspace of F^* which contains H and suppose that the $\sigma(F^*, F)$-closure of each $\sigma(F^*, F)$-bounded subset of G which is essentially separable for the dual pair (F^*, F) is contained in G. Since $M^{\bullet}$ is $\sigma(F^*, F)$-complete, $G \cap M^{\bullet}$ has the same property. Reasoning as before, we have that $H^{\delta_1} \subseteq H^{\delta_2}$ so that $H^{\delta_1} = H^{\delta_2}$ and the proof is complete.

The next two results are analogues of A, A_1, A_2 of §3 and B, B_1 of §4 of [1]. The proofs make use of some of the techniques of the corresponding results in [1]. Our results characterize Mackey spaces which are δ-barrelled and they give a precise description of the range spaces in our closed graph theorem in which a δ-barrelled Mackey space is the domain space.

<u>Theorem 4.1.3</u>

(i) A linear mapping with a closed graph of a δ-barrelled Mackey space into an infra-δ-space is continuous.

(ii) Let F be a separated locally convex space with the property that any linear mapping with a closed graph of a δ-barrelled Mackey space into F is continuous. Then F is an infra-δ-space.

(iii) Let E be a Mackey space with the property that any linear mapping with a closed graph of E into an infra-δ-space is continuous. Then E is δ-barrelled.

<u>Proof</u>

(i) Let $t : E \to F$ be a linear mapping with a closed graph of a δ-barrelled Mackey space E into an infra-δ-space F. Let $t^* : F^* \to E^*$ be the algebraic transpose of t. Then t^* is continuous under $\sigma(F^*, F)$ and $\sigma(E^*, E)$. If $H = t^{*-1}(E') \cap F'$, then trivially $H \subseteq t^{*-1}(E')$ and also if A is any $\sigma(F^*, F)$-bounded essentially separable subset of $t^{*-1}(E')$, by Lemma 2.1.2, we have that $t^*(A)$ is a $\sigma(E', E)$-bounded essentially separable subset of E'. Since E is δ-barrelled, $t^*(A)$ is $\sigma(E', E)$-relatively compact. Hence the $\sigma(E', E)$-closure of $t^*(A)$ is $\sigma(E^*, E)$-closed and consequently t^* maps the $\sigma(F^*, F)$-closure of A into E'. Thus the $\sigma(F^*, F)$-closure of A is contained in $t^{*-1}(E')$. It therefore follows that $t^{*-1}(E')$ is one of the subspaces of F^* considered in the intersection which determines H^δ. Hence $H^\delta \subseteq t^{*-1}(E')$ so that $H^\delta \cap F' \subseteq t^{*-1}(E') \cap F'$ i.e. $(t^{*-1}(E') \cap F')^\delta \cap F' \subseteq t^{*-1}(E') \cap F'$. It follows therefore that $t^{*-1}(E') \cap F' = (t^{*-1}(E') \cap F')^\delta \cap F' = F'$ since F is an infra-δ-space and $t^{*-1}(E') \cap F'$ is a $\sigma(F', F)$-dense vector subspace of F'. Consequently t is weakly continuous and so by [43, Chapter III, Proposition 14], t is continuous since E is a Mackey space.

(ii) Let H be any $\sigma(F', F)$-dense vector subspace of F'. Since H separates the points of F so also does H^δ and so (F, H^δ) is a dual pair. From the construction of H^δ we see that every $\sigma(H^\delta, F)$-bounded essentially separable subset of H^δ is $\sigma(H^\delta, F)$-relatively compact and so F is δ-barrelled under $\tau(F, H^\delta)$. We now consider the identity mapping $i : (F, \tau(F, H^\delta)) \to (F, \xi)$, where ξ is the initial topology of F. The transpose $i' : F' \to F^*$ of i is just the natural injection of F' into F^* and so $i'^{-1}(H^\delta) = H^\delta \cap F' \supseteq H$. Consequently, i has a closed graph [43, Chapter VI, Lemma 4]. Hence by the hypothesis, i is continuous and so $H^\delta \cap F' = F'$. Thus F is an infra-δ-space.

(iii) As we have seen in Example 4.1.1(a), a Fréchet space of dimension at most c is a δ-space and so is an infra-δ-space. Hence in particular every Banach space of dimension at most c is an infra-δ-space. Thus if $t : E \to F$ is a linear mapping with a closed graph of E into an arbitrary Banach space of dimension at most c, by the hypothesis, t is continuous. Hence by Theorem 3.1.2, we have that E is δ-barrelled and the proof is complete.

The following is effectively established in the proof of (i) above.

Corollary

A linear mapping with a closed graph of a separated δ-barrelled space into an infra-δ-space is weakly continuous.

Theorem 4.1.4

(i) A linear mapping with a closed graph of a δ-space onto a δ-barrelled Mackey space is open.

(ii) Let F be a separated locally convex space with the property that any linear mapping with a closed graph of F onto a δ-barrelled Mackey space is open. Then F is a δ-space.

Proof

(i) The proof of (i) uses standard techniques and so we give a sketch of it. Let $t : E \to F$ be a linear mapping with a closed graph of a δ-space E onto a δ-barrelled Mackey space F. Let $N = t^{-1}(\{0\})$. We express t as $t = soq$, where $q : E \to E/_N$ is the quotient map and s is a one-to-one linear mapping of E/N onto F. We observe that q is open since E/N has its quotient topology and that $s^{-1} : F \to E/_N$ is a linear mapping with a closed graph (see for example [43, Chapter VI, proof of Proposition 10 part (2)]. By Theorem 4.1.2, we have that E/N is an infra-δ-space and so it follows from Theorem 4.1.3(i) that s^{-1} is continuous. Hence s and consequently t are open.

(ii) Let H be a vector subspace of F'. We note first that since $(F/_{H^\circ})'$ is the $\sigma(F', F)$-closure of H and since $H \subseteq H^\delta \subseteq H^{\circ\circ}$, it follows that $(F/_{H^\circ}, H^\delta)$ is a dual pair. Let $q : F \to F/_{H^\circ}$ be the quotient map of F onto $F/_{H^\circ}$ under the topology $\tau(F/_{H^\circ}, H^\delta)$. Reasoning as in the proof of Theorem 4.1.3(ii), we have that $F/_{H^\circ}$ is δ-barrelled under $\tau(F/_{H^\circ}, H^\delta)$. Also q has a closed graph since its transpose is the natural injection of H^δ into F^*. By the hypothesis therefore, q is open and so $\tau(F/_{H^\circ}, H^\delta)$ must be finer than the quotient topology on $F/_{H^\circ}$ for which the dual is the $\sigma(F', F)$-closure $H^{\circ\circ}$ of H. Thus we have that $H^{\circ\circ} \subseteq H^\delta$. The result now follows when we observe that $H^\delta \cap F'$ is contained in the $\sigma(F', F)$-closure of H.

It is interesting to note that in Theorem 4.1.3 and Theorem 4.1.4, it is not enough to have an arbitrary separated δ-barrelled space. The following example illustrates this fact.

<u>Example 4.1.2</u>

Let E be the Hilbert space $\ell_2(\Gamma)$ where $|\Gamma| > c$. As we have noted in Example 3.2.1(ii), the topology $\delta(\ell_2(\Gamma), \ell_2(\Gamma))$ is strictly coarser than the norm topology of $\ell_2(\Gamma)$. For any topology of the dual pair $(\ell_2(\Gamma), \ell_2(\Gamma))$, a point of closure of any vector subspace H of $\ell_2(\Gamma)$ is the limit of a sequence in H and so $\overline{H} \subseteq H^\delta$. It therefore follows that $\ell_2(\Gamma)$ is a δ-space for any such topology. Consider the identity mapping i of $\ell_2(\Gamma)$ under $\delta(\ell_2(\Gamma), \ell_2(\Gamma))$ onto $\ell_2(\Gamma)$ with its norm topology. It is clear that i has a closed graph. However i is not continuous and its inverse is also not open even though it has a closed graph being continuous.

We now give the promised example of an infra-δ-space which is not weakly t-polar. For this we adapt ideas from Sulley's example in [50].

Example 4.1.3

Let E be the real Banach space of all families $x = (x_\alpha)_{\alpha\in\mathbb{N}\times\mathbb{N}}$ of real numbers which converge to zero with respect to the filter of complements of finite subsets of $\mathbb{N}\times\mathbb{N}$ with norm $\|x\| = \sup_{\alpha\in\mathbb{N}\times\mathbb{N}} |x_\alpha|$ ([6, Chapter IV, page 121, Exercise 14]).

For every $x = (x_\alpha)_{\alpha\in\mathbb{N}\times\mathbb{N}}$ in E and for every $\epsilon > 0$, there exists a finite subset ϕ of $\mathbb{N}\times\mathbb{N}$ such that $\|x_\alpha\| \le \epsilon$ if $\alpha \notin \phi$. For each $\alpha \in \phi$, we choose rational q_α such that $|x_\alpha - q_\alpha| \le \epsilon$ and put $q_\alpha = 0$ if $\alpha \notin \phi$. Then $(q_\alpha)_{\alpha\in\mathbb{N}\times\mathbb{N}} \in E$ and we have $\|(x_\alpha)_{\alpha\in\mathbb{N}\times\mathbb{N}} - (q_\alpha)_{\alpha\in\mathbb{N}\times\mathbb{N}}\| \le \epsilon$. Hence $\{(x_\alpha)_{\alpha\in\mathbb{N}\times\mathbb{N}} : x_\alpha \in \mathbb{Q}, \{\alpha : x_\alpha \ne 0\}$ is finite$\}$ is a countable dense subset of E and so E is separable in its norm topology.

If E' is the dual of E, then as shown in [6, Chapter IV, page 121, Ex. 14], there exists a proper $\sigma(E', E)$-dense vector subspace M of E' such that the $\sigma(E', E)$-closure of the intersection with M of the closed unit ball of E' generates a proper subspace of E'. Consider the dual pair (E, M). It is shown in [50] that $(E, \sigma(E, M))$ is an infra-s-space which is not weakly t-polar.

We show that E is an infra-δ-space for any topology of the dual pair (E, M). We note first that since, in the norm topology, E is an infinite dimensional separable Banach space and so has dimension c, every linear mapping with a closed graph of a δ-barrelled space into E is necessarily continuous (Theorem 3.1.2). It then follows from Theorem 4.1.3(ii) that E is an infra-δ-space for the norm topology.

Since M is $\sigma(E', E)$-dense, we now use Theorem 4.1.1 to conclude that E is an infra-δ-space for any topology of the dual pair (E, M). Thus an infra-δ-space need not be weakly t-polar.

It follows from Theorem 4.1.3(ii), Example 3.1.1(a) and Adasch's characterization of barrelled spaces [1, §3, A_1, A_2], that the infra-δ-spaces form a proper subclass of the infra-s-spaces. We give next a specific example of an infra-s-space which is not an infra-δ-space.

Example 4.1.4

Let $E = \mathbb{R}^{(M)}$, where $|M| > c$,

$$E' = \{(\xi_\mu)_{\mu \in M} \in \mathbb{R}^M : |\{\mu : \xi_\mu \neq 0\}| \leq c\} ,$$

$$B = \{(\xi_\mu)_{\mu \in M} \in \mathbb{R}^M : |\xi_\mu| \leq 1 \text{ for all } \mu \in M\} .$$

Using the notations of Section 2.1, we let $F = \mathcal{B}(E, B)$, the Banach space constructed from B. Then F is an infra-s-space since it is a Banach space and therefore is B-complete. As we have seen in Example 3.1.1(a), E is δ-barrelled under $\tau(E, E')$. The natural embedding j of the δ-barrelled Mackey space $(E, \tau(E, E'))$ into F has a closed graph since its transpose j' is the natural injection of F' (the linear span of B), into $E^* = \mathbb{R}^M$ and $j'^{-1}(E')$ is $\sigma(F', F)$-dense in F'. But j is not continuous since $j'(F') \not\subseteq E'$.

The associated δ-barrelled topology of a locally convex space is defined in Section 3.4 in a way similar to the definitions in [12] and [1] of the associated barrelled topology. It is shown in [12] and [3] that an infra-s-space is complete in its associated barrelled topology. As we have seen, the infra-δ-spaces form a proper subclass of the class of the infra-s-spaces. Thus an infra-δ-space is also complete in its associated barrelled topology. We show next that an infra- δ-space is in fact complete in its associated δ-barrelled topology which is necessarily coarser than the associated barrelled topology. Further we given an example, later, of an infra-δ-space for which the associated barrelled topology and the associated δ-barrelled topology are not even topologies of the same dual pair. This will then show that the completeness of an infra- δ-space in its associated δ-barrelled topology is in fact a genuine improvement on the corresponding earlier result.

The method used in the proof of Theorem 4.1.5 is a modification of the proof of Theorem 1.5 of [12] and so we give just a sketch of it.

<u>Theorem 4.1.5</u>

Let (F, η) be an infra-δ-space. Then (F, δ_η), that is F under its associated δ-barrelled topology, is complete.

<u>Proof</u>

Suppose on the contrary that (F, δ_η) is not complete. Let $\hat{F}$ be the completion of (F, δ_η) and let $x_o \in \hat{F}\backslash F$. If G is the linear span of x_o then as in [12, Theorem 1.5], the restriction $q|_F$ of the quotient map $q : \hat{F} \to \hat{F}/_G$ to F is injective. Define $f : F \to q(F)$ by $f(x) = q|_F(x)$. Then f is a bijection and since F is dense in $\hat{F}$, by the lemma in [49, page 275], f is continuous and almost open when F has the topology δ_η and $q(F)$ has the topology ξ induced from the quotient topology of $\hat{F}/_G$ (considering the extension of δ_η on $\hat{F}$). Hence by Theorem 3.1.4 we must have that $(q(F), \xi)$ is a δ-barrelled space.

Reasoning as in the proof of (1.7) of [12] and using Lemma 3.4.1 in place of (1.2), we have that the graph of f is closed in $(F, \eta) \times (q(F), \xi)$. Since closedness for convex sets is the same for all topologies of a given dual pair we have immediately that $f^{-1} : (q(F), \tau) \to (F, \eta)$ is continuous, where τ is the Mackey topology corresponding to ξ (Remark (iv) after Theorem 3.1.1, Theorem 4.1.3). By Theorem 3.4.1, we have that $f^{-1} : (q(F), \tau) \to (F, \delta_\eta)$ is continuous and so weakly continuous. Consequently $f : (F, \delta_\eta) \to (q(F), \tau)$ is weakly open (also open). Since a separated quotient of a weak topology is a weak topology and a weak topology induces a weak topology on a vector subspace, we may again apply the lemma in [49, page 275] (when $\hat{F}$ has its weak topology) to deduce that $F \cap G$ is (weakly) dense in G which is impossible since $F \cap G = \{0\}$. This contradiction establishes the result.

Example 4.1.5

As we have in Example 4.1.1(c), define E, E' by $E = \mathbb{R}^{(M)}$, where $|M| > c$, $E' = \{(\xi_\mu)_{\mu \in M} \in \mathbb{R}^M : |\{\mu : \xi_\mu \neq 0\}| \leq \aleph_0\}$.

Then E is an infra-δ-space for any topology of the dual pair (E, E'). We observe that for any subset B of $\mathbb{R}^M$ which is a product of symmetric intervals, $B \cap E'$ is $\sigma(\mathbb{R}^M, \mathbb{R}^{(M)})$-dense in B and also that every bounded subset of $\mathbb{R}^M$ is contained in such a product of intervals. It then follows from these observations that the associated barrelled topology for any topology of the dual pair (E, E') is $\tau(\mathbb{R}^{(M)}, \mathbb{R}^M) = \beta(\mathbb{R}^{(M)}, \mathbb{R}^M)$.

Now let $F' = \{(\xi_\mu)_{\mu \in M} \in \mathbb{R}^M : |\{\mu : \xi_\mu \neq 0\}| \leq c\}$. We have seen in Examples 3.1.1(a) and 3.2.2 that E has δ-barrelled topologies for the dual pair (E, F'). Clearly, the associated δ-barrelled topology $\delta_{\sigma(E, E')}$ is coarser than $\delta(E, F')$ (Section 3.2).

On the other hand, if B is a $\sigma(F', E)$-bounded essentially separable subset of F', then $|\text{supp } B| \leq c$ (Example 2.2.1). Since B is bounded, it follows that B is contained in a subset $\prod_{\mu \in M} I_\mu$ of F', where for each $\mu \in M$, I_μ is of the form $[-a_\mu, a_\mu]$ and $a_\mu = 0$ if $\mu \notin \text{supp } B$. But $(\prod_{\mu \in M} I_\mu) \cap E'$ is $\sigma(F', E)$-dense in $\prod_{\mu \in M} I_\mu$ and it is also a $\sigma(E', E)$-bounded essentially separable set since $\prod_{\mu \in M} I_\mu$ is a separable subset of the algebraic dual. Thus $(\prod_{\mu \in M} I_\mu) \cap E'$ must be $\delta_{\sigma(E, E')}$-equicontinuous. It follows that $\prod_{\mu \in M} I_\mu$ and consequently B are contained in the dual of E under $\delta_{\sigma(E, E')}$ and are $\delta_{\sigma(E, E')}$-equicontinuous. This shows that $\delta_{\sigma(E, E')}$ is finer than $\delta(E, F')$. Thus $\delta(E, F') = \delta_{\sigma(E, E')}$.

Since $F' \neq \mathbb{R}^M$, the associated barrelled topology and the associated δ-barrelled topology of the space $(E, \sigma(E, E'))$ are not topologies of the same dual pair.

CHAPTER V

SOME RELATED TOPICS

We devote this chapter to the discussion of some topics that are related to the notions of essential separability and δ-barrelledness which we have already met in the previous chapters. First we look at the domain spaces for the closed graph theorem in the unusual case in which the range space is not necessarily complete. Then we consider an extension of the idea of δ-barrelled spaces to general topological vector spaces. Finally we are concerned with the problem of replacing c with an arbitrary infinite cardinal where the significant factor is now density character rather than dimension. In this setting we obtain a general closed graph theorem (Theorem 5.3.1) which includes our earlier Theorem 3.1.2 and Kalton's closed graph theorem [25, 2.6] as special cases and also a characterization of Valdivia's γ-barrelled spaces.

5.1 Incomplete Range Spaces

The topological vector spaces that serve as range spaces for the closed graph theorem usually have something to do with one kind of completeness or another. It is natural to ask what happens when the range space is not assumed to be complete. The situation when the range space is an arbitrary normed space has been considered by V. Eberhardt [14] and in [15], V. Eberhardt and W. Roelcke discuss the case when a metrizable locally convex space serves as the range space (see also [11]). In this section we discuss the domain spaces for a closed graph theorem in which the range space is a normed (respectively metrizable locally convex) space with dimension at most c. The following definitions are analogous to the definitions of the GN- and GM- spaces in [14] and [15] respectively.

Definition 5.1.1

We define the classes $\mathcal{N}_c$ and $\mathcal{M}_c$ of separated locally convex spaces as follows: $E \in \mathcal{N}_c$ (respectively $\mathcal{M}_c$) if whenever $t : E \to F$ is a linear mapping with a closed graph of E into a normed (respectively metrizable locally convex) space F with dimension at most c, t is necessarily continuous.

We note that since a Banach space is trivially a normed space and a metrizable locally convex space, elements of $\mathcal{N}_c$ and $\mathcal{M}_c$ are necessarily δ-barrelled (Theorem 3.1.2).

We now give descriptions of spaces in $\mathcal{N}_c$ and $\mathcal{M}_c$ in terms of the concepts of essential separability and δ-topology. The results are similar to (1.2) of [14] and (2.3) of [15] but the proofs are rather different. Note that in Theorems 5.1.1 and 5.1.2 below, condition (a) implies that there are δ-barrelled topologies for the dual pair (E, E') so that (b) is meaningful.

Theorem 5.1.1

Let (E, ξ) be a separated locally convex space. Then $E \in \mathcal{M}_c$ if and only if

(a) for any sequence (B_n) of non-empty $\sigma(E', E)$-bounded essentially separable sets, the $\sigma(E', E)$-closed linear span of $\bigcup_{n=1}^{\infty} B_n$ is $\sigma(E', E)$-complete, and

(b) ξ is finer than $\delta(E, E')$.

Proof

Suppose first that $E \in \mathcal{M}_c$. Let (B_n) be any sequence of $\sigma(E', E)$-bounded essentially separable sets. For each n, let C_n be the $\sigma(E', E)$-closed absolutely convex envelope of B_n and let G be the linear span of $\bigcup_{n=1}^{\infty} C_n$. Then $(E/_{G^o})^*$ is isomorphic to the

$\sigma(E^*, E)$-closure H of G and H is $\sigma(E^*, E)$-separable by Theorem 2.1.2 and Lemma 2.1.1. Thus by Theorem 1.5.1 the dimension of $E/_{G^\circ}$ is at most c. Also, $(E/_{G^\circ}, G)$ is a dual pair and by considering the topology of uniform convergence on the sets C_n $(n \in \mathbb{N})$ each of which is $\sigma(G, E/_{G^\circ})$-compact since E is δ-barrelled, we get that $E/_{G^\circ}$ is metrizable under $\tau(E/_{G^\circ}, G)$.

Let x^* be an arbitrary element of H and let L be the linear span of $\{x^*\} \cup G$. Then $E/_{G^\circ}$ is also metrizable under $\tau(E/_{G^\circ}, L)$ which is finer than $\tau(E/_{G^\circ}, G)$. Since the linear span of G is $\sigma(E^*, E)$-dense in L, it follows from [43, Chapter VI, Lemma 4] that the graph of the quotient map $q : E \to E/_{G^\circ}$ is closed in $(E, \xi) \times (E/_{G^\circ}, \tau(E/_{G^\circ}, L))$. Thus by the hypothesis, q is continuous and so since the transpose $q' : L \to E'$ is the natural injection, it follows in particular that $x^* \in E'$. Since $x^* \in H$ was arbitrary, this shows that $H \subseteq E'$ and establishes the necessity of (a).

As already noted, each element of $\mathcal{M}_c$ is δ-barrelled. The necessity of (b) now follows.

Conversely, suppose that (a) and (b) are satisfied and let $t : E \to F$ be a linear mapping with a closed graph of E into a metrizable locally convex space F with dimension at most c. Let (U_n) be a base of neighbourhoods of the origin in F. Then for each n, we have that U_n° is an essentially separable subset of F', since the dimension of F is at most c. Since the transpose $t' : F' \to E^*$ of t is weakly continuous it follows from Lemmas 2.1.1 and 2.1.2 and the Corollary to Lemma 2.1.3 that $(t'(U_n^\circ) \cap E')$ is a sequence of non-empty $\sigma(E', E)$-bounded essentially separable sets. If G is the $\sigma(E', E)$-closed linear span of $\bigcup_{n=1}^{\infty}(t'(U_n^\circ) \cap E')$, then by (a) G is $\sigma(E', E)$-complete and therefore $\sigma(E^*, E)$-closed. Now,

$t'^{-1}(E') = \bigcup_{n=1}^{\infty} t'^{-1}(t'(U_n^{\circ}) \cap E')$ and $t'^{-1}(E')$ is $\sigma(F', F)$-dense in F'. It now follows that $t'(F') \subseteq G \subseteq E'$. Hence t' is weaky continuous and applying Lemma 2.1.2 again we have that for each n, $t'(U_n^{\circ})$ is a $\sigma(E', E)$-bounded essentially separable set and so is equicontinuous by (b). Thus t is continuous as required.

The proof of Theorem 5.1.2 below is similar to that of Theorem 5.1.1 and so it will be omitted.

Theorem 5.1.2

Let (E, ξ) be a separated locally convex space. Then $(E, \xi) \in \mathcal{N}_c$ if and only if

(a) the $\sigma(E', E)$-closed linear span of each non-empty $\sigma(E', E)$-bounded essentially separable set is $\sigma(E', E)$-complete, and

(b) ξ is finer than $\delta(E, E')$.

Corollary

Let $E \in \mathcal{N}_c$. Then $\delta(E, E')$ is the topology of uniform convergence on the $\sigma(E', E)$-bounded separable sets.

Proof

Let η be the topology on E of uniform convergence on the $\sigma(E', E)$-bounded separable sets. Clearly η is coarser than $\delta(E, E')$. Let A be any non-empty $\sigma(E', E)$-bounded essentially separable set and let H be the $\sigma(E', E)$-closed linear span of A. Since H is $\sigma(E', E)$-complete it follows from Theorem 2.1.2 that there is a $\sigma(E', E)$-bounded separable subset B of H such that $A \subseteq B$. This implies that η is finer than $\delta(E, E')$ and so we have $\eta = \delta(E, E')$.

The following theorem gives some basic permanence properties of the classes $\mathcal{N}_c$ and $\mathcal{M}_c$.

<u>Theorem 5.1.3</u>

(a) An inductive limit of spaces in $\mathcal{N}_c$ (respectively $\mathcal{M}_c$) belongs to $\mathcal{N}_c$ (respectively $\mathcal{M}_c$).

(b) The product of any collection of members of $\mathcal{N}_c$ belongs to $\mathcal{N}_c$.

(c) If $E \in \mathcal{N}_c$ (respectively $\mathcal{M}_c$) and F is the completion of E , then $F \in \mathcal{N}_c$ (respectively $\mathcal{M}_c$).

<u>Proof</u>

(a) Uses the usual techniques (cf. proof of Corollary 1 of Theorem 3.1.2).

(b) Is obtained by Eberhardt's technique in [14, 1.3].

(c) We establish the result for the class $\mathcal{N}_c$. The proof for the class $\mathcal{M}_c$ follows the same pattern.

Suppose that $E \in \mathcal{N}_c$. Since E is δ-barrelled, so also is F and the same subsets of E' are bounded and essentially separable for the dual pairs (E', E) and (E', F) (Corollary 3 of Theorem 3.1.2, Corollary to Theorem 2.1.1 and Corollary 1 of Theorem 2.1.7). Let B be a non-empty bounded essentially separable subset of E' and let H be the $\sigma(E', E)$-closed linear span of B . Since $E \in \mathcal{N}_c$, H is $\sigma(E', E)$-complete (Theorem 5.1.2). Consequently $(E/_{H}o)^* = H$ and since each bounded subset of H is essentially separable and since E is δ-barrelled, the quotient topology on $E/_{H}o$ is $\tau(E/_{H}o , H) = \beta(E/_{H}o , H)$, under which $E/_{H}o$ is complete [43, Chapter III, supplement (2)].

If $z \in F$, then z is $\sigma(E', E)$-continuous on each $\sigma(E', E)$-bounded essentially separable set and therefore on each $\sigma(E', E)$-bounded subset of H. This implies that the restriction of z to H belongs to the completion of the quotient space $E/_{H^o}$ and therefore to $E/_{H^o}$.

It follows that the topologies $\sigma(E', E)$ and $\sigma(E', F)$ coincide on H, which must therefore be the $\sigma(E', F)$-closed linear span of B and $\sigma(E', F)$-complete. The result now follows since F is δ-barrelled.

Example 5.2.1

(i) It is clear from the definitions that $\mathcal{N}_c$ contains $\mathcal{M}_c$. In fact $\mathcal{M}_c$ is properly contained in $\mathcal{N}_c$. For example let $E = \mathbb{R}^{\mathbb{N}}$ and let $E' = \mathbb{R}^{(\mathbb{N})}$. We note that the product topology on $\mathbb{R}^{\mathbb{N}}$ is $\sigma(\mathbb{R}^{\mathbb{N}}, \mathbb{R}^{(\mathbb{N})}) = \tau(\mathbb{R}^{\mathbb{N}}, \mathbb{R}^{(\mathbb{N})})$ and that $\mathbb{R}^{\mathbb{N}}$ is barrelled and so is in particular δ-barrelled. Since each $\sigma(\mathbb{R}^{(\mathbb{N})}, \mathbb{R}^{\mathbb{N}})$-bounded set is contained in $\bigoplus_{n\in\mathbb{N}} T_n$, where $T_n = \mathbb{R}$ for finitely many n and $T_n = \{0\}$ otherwise, it is clear that the $\sigma(\mathbb{R}^{(\mathbb{N})}, \mathbb{R}^{\mathbb{N}})$-closed linear span of any $\sigma(\mathbb{R}^{(\mathbb{N})}, \mathbb{R}^{\mathbb{N}})$-bounded set is $\sigma(\mathbb{R}^{(\mathbb{N})}, \mathbb{R}^{\mathbb{N}})$-complete. Hence by Theorem 5.1.2, we have that $(\mathbb{R}^{\mathbb{N}}, \sigma(\mathbb{R}^{\mathbb{N}}, \mathbb{R}^{(\mathbb{N})})) \in \mathcal{N}_c$. However, $(\mathbb{R}^{\mathbb{N}}, \sigma(\mathbb{R}^{\mathbb{N}}, \mathbb{R}^{(\mathbb{N})})) \notin \mathcal{M}_c$, for if we define the sequence (B_m) by $B_m = \{(\delta_{mn})_{n\in\mathbb{N}}\}$, then (B_m) is a sequence of $\sigma(\mathbb{R}^{(\mathbb{N})}, \mathbb{R}^{\mathbb{N}})$-bounded essentially separable sets and $\bigcup_{m=1}^{\infty} B_m$ spans $\mathbb{R}^{(\mathbb{N})}$. Thus the $\sigma(\mathbb{R}^{(\mathbb{N})}, \mathbb{R}^{\mathbb{N}})$-closed linear span of $\bigcup_{m=1}^{\infty} B_m$ is $\mathbb{R}^{(\mathbb{N})}$. But $\mathbb{R}^{(\mathbb{N})}$ is not $\sigma(\mathbb{R}^{(\mathbb{N})}, \mathbb{R}^{\mathbb{N}})$-complete, for it is weakly dense in $(\mathbb{R}^{\mathbb{N}})^*$ and $\mathbb{R}^{(\mathbb{N})} \neq (\mathbb{R}^{\mathbb{N}})^*$ since there exists a non-zero element of $(\mathbb{R}^{\mathbb{N}})^*$ which annihilates $\mathbb{R}^{(\mathbb{N})} \subseteq \mathbb{R}^{\mathbb{N}}$. But such a linear form cannot be represented by an element of $\mathbb{R}^{(\mathbb{N})}$. Thus by Theorem 5.1.1 we have that $(\mathbb{R}^{\mathbb{N}}, \sigma(\mathbb{R}^{\mathbb{N}}, \mathbb{R}^{(\mathbb{N})})) \notin \mathcal{M}_c$.

(ii) As we have already noted, it follows immediately from the definition that every member of $\mathcal{N}_c$ is a separated δ-barrelled space. On the other hand, not every δ-barrelled space is in $\mathcal{N}_c$. Take for example any Banach space F with dimension c. Trivially, F is separated and δ-barrelled. However $F \notin \mathcal{N}_c$, for the closed unit ball B of F' is a $\sigma(F', F)$-bounded essentially separable set whose $\sigma(F', F)$-closed linear span cannot be $\sigma(F', F)$-complete, the linear span of B being the whole of F' [43, Chapter III, supplement (2)].

(iii) It is not difficult to see that the separated locally convex space (E, ξ) of Example 3.2.2 satisfies the conditions of Theorem 5.1.1. Thus the space (E, ξ) provides us with an example of an element of $\mathcal{M}_c$ which is not countably barrelled (c.f. [15,2.5]).

The next result is an analogue of (2.4) of [14].

<u>Theorem 5.1.4</u>

Let $E \in \mathcal{N}_c$ (respectively $\mathcal{M}_c$) and let G be a vector subspace of E. Then $G \in \mathcal{N}_c$ (respectively $\mathcal{M}_c$) if and only if G is δ-barrelled in the topology induced by $\delta(E, E')$.

<u>Proof</u>

We give the proof for $\mathcal{N}_c$. The proof for $\mathcal{M}_c$ follows the same pattern.

Suppose that $G \in \mathcal{N}_c$. We observe first that the dual of G is (up to isomorphism) $E'/_{G^o}$. Let $(\bar{x}'_n)$ be any $\sigma(E'/_{G^o}, G)$-bounded sequence. Since $\{\bar{x}'_n : n \in \mathbb{N}\}$ is an equicontinuous subset of G', there is an equicontinuous subset C of E' such that $\{\bar{x}'_n : n \in \mathbb{N}\} \subseteq q(C)$, where $q : E' \to E/_{G^o}$ is the quotient map. For each $n \in \mathbb{N}$ choose $y'_n \in C$ such that $q(y'_n) = \bar{x}'_n$.

Then $\{y'_n : n \in \mathbb{N}\}$ is equicontinuous under $\delta(E, E')$, from which it follows using the Corollary to Theorem 5.1.2, that $\delta(E, E')$ induces $\delta(G, G')$ on G.

Conversely, suppose that G is δ-barrelled under the topology η induced by $\delta(E, E')$. Let A be any $\sigma(E'/_{G^O}, G)$-bounded essentially separable set, let B be the $\sigma(E'/_{G^O}, G)$-closed absolutely convex envelope of A and let H be the linear span of B. Let $q : E' \to E'/_{G^O}$ be the quotient map. Since B is η-equicontinuous, there exists a $\sigma(E', E)$-compact essentially separable absolutely convex set B_1, such that $B \subseteq q(B_1)$. Put $B_2 = q^{-1}(B) \cap B_1$ so that B_2 has the same properties as B, and further $q(B_2) = B$. Let F be the linear span of B_2; clearly $q(F) = H$. Consider the linear mapping $\phi : G/_{H^O} \to E/_{F^O}$ defined by $\phi(x + H^O) = x + F^O$. The mapping ϕ is well-defined since if $x \in H^O$ and $x' \in F$ then

$$\langle x, x'\rangle = \langle q'(x), x'\rangle = \langle x, q(x')\rangle = 0 ,$$

i.e. $x \in F^O$. A similar argument shows that ϕ is $1 - 1$. Also since q maps F^{OO} into H^{OO} (bipolars in E', $E'/_{G^O}$ respectively) it is easy to see that $\phi' = q|_{F^{OO}}$ and consequently that ϕ is continuous under $\sigma(G/_{H^O}, H^{OO})$ and $\sigma(E/_{F^O}, F^{OO})$.

If $L = \phi(G/_{H^O})$, then $\phi^{-1} : L \to G/_{H^O}$ is trivially continuous since $\sigma(E/_{F^O}, F^{OO})$ induces $\sigma(L, L^*)$ on L (F^{OO} being $\sigma(E', E)$-complete). Thus $\sigma(G/_{H^O}, H^{OO})$ must be $\sigma(G/_{H^O}, (G/_{H^O})^*)$ and so H^{OO} is $\sigma(G', G)$-complete. Thus the condition (a) of Theorem 5.1.2 is satisfied. It is clear that (b) of the theorem is also satisfied and consequently $G \in \mathcal{N}_c$.

Corollary

If $E \in \mathcal{N}_c$ (respectively $\mathcal{M}_c$) and G is a vector subspace of E with countable codimension, then $G \in \mathcal{N}_c$ (respectively $\mathcal{M}_c$) .

Proof

By Theorem 3.1.6, we have that G is δ-barrelled under a topology induced by any δ-barrelled topology on E . In particular G is δ-barrelled under $\delta(E, E')|_G$. The result now follows from the theorem.

It has been shown [14, 2.5] that the separated GN-spaces are, up to isomorphism, precisely the barrelled subspaces of products of spaces with the finest locally convex topologies. The following is the corresponding result for $\mathcal{N}_c$.

Theorem 5.1.5

Let G be a subspace of a product $\pi_{\alpha \in A} E_\alpha$ of separated locally convex spaces such that

(i) $\dim E_\alpha \leq c \quad (\alpha \in A)$,

(ii) E_α has its finest locally convex topology $(\alpha \in A)$,

(iii) G is δ-barrelled (in the topology induced by the product topology on $\pi_{\alpha \in A} E_\alpha$) .

Then $G \in \mathcal{N}_c$. Conversely each element E of $\mathcal{N}_c$ is isomorphic under $\delta(E, E')$ to such a G .

Proof

Trivially each $E_\alpha \in \mathcal{N}_c$ and E_α has the topology $\delta(E_\alpha, E'_\alpha)$. By Theorem 5.1.3(b), $\pi_{\alpha \in A} E_\alpha \in \mathcal{N}_c$ and its product topology is $\delta(\pi_{\alpha \in A} E_\alpha \; \sum_{\alpha \in A} E'_\alpha)$ (see Section 3.3). It follows from Theorem 5.1.4 that $G \in \mathcal{N}_c$.

Let $E \in \mathcal{N}_c$ and let $\mathcal{B}$ be the family of all non-empty $\sigma(E', E)$-bounded essentially separable sets. For each $B \in \mathcal{B}$ let $H(B)$ be the $\sigma(E', E)$-closed linear span of B and let $E(B) = E/_{H(B)^o}$. Since $H(B)$ is $\sigma(H(B), E(B))$-complete (Theorem 5.1.2), the topology $\tau(E(B), H(B))$ is the finest locally convex topology on $E(B)$ and $E(B)$ is barrelled under $\tau(E(B), H(B))$. Since B is essentially separable, $\dim E(B) \leq c$ (Theorem 2.1.3). We note also that each $\sigma(H(B), E(B))$-bounded set is essentially separable so that $\tau(E(B), H(B))$ coincides with $\delta(E(B), H(B))$.

We define $t : E \to \prod_{B\in\mathcal{B}} E(B)$ by $t(x) = (q_B(x))_{B\in\mathcal{B}}$ where $q_B : E \to E(B)$ is the quotient map. Then t is clearly one-to-one and linear. Let t' be the transpose of t. Then $t'((x'_B)_{B\in\mathcal{B}}) = \sum_{B\in\mathcal{B}} x'_B$ for all $(x'_B)_{B\in\mathcal{B}} \in \sum_{B\in\mathcal{B}} H(B)$ so that $t'(\sum_{B\in\mathcal{B}} H(B)) \subseteq E'$ and since each $\sigma(\sum_{B\in\mathcal{B}} H(B), \prod_{B\in\mathcal{B}} E(B))$-bounded set is essentially separable, it follows that t is continuous under $\delta(E, E')$ and the product of the topologies $\tau(E(B), H(B))$ which, as above is $\delta(\prod_{B\in\mathcal{B}} E(B), \sum_{B\in\mathcal{B}} H(B))$.

Let t^{-1} be the inverse of t on $t(E)$. For each $B \in \mathcal{B}$, $q_B \circ t^{-1} = p_B|_{t(E)}$, where p_B is the projection of $\prod_{B\in\mathcal{B}} E(B)$ onto $E(B)$. Consequently t^{-1} is continuous when E has the projective limit topology of the topologies $\tau(E(B), H(B))$ by the mappings q_B $(B \in \mathcal{B})$. We show finally that this projective limit topology is $\delta(E, E')$. It has for a base of neighbourhoods of the origin all sets of the form $\bigcap_{r=1}^{n} q_B^{-1}(V_r)$ where each V_r is the polar in $E(B_r)$ of a $\sigma(H(B), E(B))$-bounded set D_r. Then $\bigcap_{r=1}^{n} q_{B_r}^{-1}(V_r) = \bigcap_{r=1}^{n} D_r^{o}$, (by [43, Chapter II, Lemma 6] since the transpose of $q|_B$ is the natural injection) and since D_r is necessarily essentially separable, we just have a base of neighbourhoods of the origin for $\delta(E, E')$.

5.2 Non-Locally Convex Spaces

Recently, there has been growing interest in topological vector spaces in which local convexity is not assumed. The notion of ultrabarrelledness which corresponds to barrelledness in this setting was first defined by W. Robertson [44]. Later, Iyahen [21, Theorem 3.2] characterized ultrabarrelled spaces as those which can serve as domain spaces for a closed graph theorem in which the range space is an arbitrary complete metric linear space. He has also defined and discussed countably ultrabarrelled spaces in [22].

In this section we introduce the class of δ-ultrabarrelled spaces whose relationship with ultrabarrelled spaces is similar to that between δ-barrelled and barrelled spaces. Many of the properties of δ-barrelled spaces are possessed in a suitable form by δ-ultrabarrelled spaces. In particular, one can always define a coarser linear topology on a δ-ultrabarrelled space which is both δ-barrelled and countably ultrabarrelled.

Recall that an ultrabarrel in a topological vector space E is a closed balanced subset B for which there exists a sequence (B_n) of balanced absorbent subsets of E such that $B_1 + B_1 \subseteq B$ and $B_{n+1} + B_{n+1} \subseteq B_n$ for each $n \in \mathbb{N}$. The sequence (B_n) is called a defining sequence for B[21, §3].

Definition 5.2.1

Let E be a topological vector space and let U be an ultra-barrel in E . We say that U is a <u>δ-ultrabarrel</u> if there is a defining sequence (U_n) for U such that the dimension of $E/\bigcap_{n=1}^{\infty} U_n$ is at most c . The space E is said to be <u>δ-ultrabarrelled</u> if every

δ-ultrabarrel in E is a neighbourhood of the origin.

Let E be a locally convex space. As is pointed out in [21, §3], if B is a barrel in E, then B is an ultrabarrel with $(2^{-n}B)$ as a defining sequence. Thus if B is a δ-barrel in E, then $(2^{-n}B)$ is a defining sequence for B and $\bigcap_{\lambda>0}\lambda B = \bigcap_{n=1}^{\infty} 2^{-n}B$ so that $\dim E/\bigcap_{n=1}^{\infty}(2^{-n}B) \leq c$. It therefore follows that every δ-barrel is a δ-ultrabarrel. Consequently a locally convex δ-ultrabarrelled space is necessarily δ-barrelled.

The following theorem gives a useful equivalent definition of a δ-ultrabarrel.

<u>Theorem 5.2.1</u>

Let E be a topological vector space. An ultrabarrel B is a δ-ultrabarrel if and only if B has a defining sequence (B_n) such that $(\bigcap_{n=1}^{\infty} B_n)^{\bullet}$ is $\sigma(E^*, E)$-separable.

<u>Proof</u>

This follows immediately from Theorem 1.5.1 and the fact that $(E/\bigcap_{n=1}^{\infty} B_n)^* = (\bigcap_{n=1}^{\infty} B_n)^{\bullet}$.

We now give a characterization of δ-ultrabarrelled spaces which corresponds to [21, Theorem 3.2]. A simple modification of the techniques of Theorem 3.1.2 gives the result (see also the proof of [21, Theorem 3.2]). We recall that the dimension of an infinite dimensional complete metrizable topological vector space is at least c (Theorem 1.5.2).

Theorem 5.2.2

A topological vector space E is δ-ultrabarrelled if and only if, whenever $t : E \to F$ is a linear mapping with a closed graph of E into a complete metrizable topological vector space of dimension at most c, then t is necessarily continuous.

The next result contains the basic permanence properties of δ-ultrabarrelled spaces. The term *-inductive limit is due to Iyahen [21, §2].

Theorem 5.2.3

(i) Any *-inductive limit of δ-ultrabarrelled spaces is δ-ultrabarrelled.

(ii) Any product of δ-ultrabarrelled spaces is also δ-ultrabarrelled.

(iii) The completion $\hat{E}$ of a δ-ultrabarrelled space E is again δ-ultrabarrelled.

Proof

(i) and (iii) follow from Theorem 5.2.2 just as in the proofs of Corollaries 1 and 3 of Theorem 3.1.2.

(ii) is proved in a way similar to [2, (2)].

As in the case of the δ-topology for a δ-barrelled space, given a δ-ultrabarrelled space E, there is a coarser related δ-ultrabarrelled topology (δu-topology) on E. For the definition of the δu-topology, we need the following result.

Theorem 5.2.4

Let (E, ξ) be a δ-ultrabarrelled space and let $\mathcal{U}$ be the collection of all δ-ultrabarrels in (E,ξ). Then $\mathcal{U}$ is a base of neighbourhoods of the origin for a vector space topology on E which is coarser than ξ .

Proof

To establish the result, it suffices to show that $\mathcal{U}$ satisfies conditions (i) - (iv) of [26, 5.1].

Let $U \in \mathcal{U}$ and let (U_n) be a defining sequence of closed sets for U . Clearly $U_1 \in \mathcal{U}$. We have $U_1 + U_1 \subseteq U$. This is (ii). Also since U is balanced $\lambda U \subseteq U$ for any scalar λ such that $|\lambda| \le 1$. (iii) is therefore satisfied. Since each element of $\mathcal{U}$ is absorbent, condition (iv) is already satisfied.

Let $U, V \in \mathcal{U}$ and let (U_n), (V_n) be defining sequences for U, V respectively such that $(\bigcap_{n=1}^{\infty} U_n)^{\bullet}$ and $(\bigcap_{n=1}^{\infty} V_n)^{\bullet}$ are $\sigma(E^*, E)$-separable. Then it follows easily that $U \cap V$ is an ultrabarrel in E with a defining sequence (W_n) defined by $W_n = U_n \cap V_n$ $(n \in \mathbb{N})$. Reasoning in a way similar to that in the proof of Lemma 3.2.2, we have that $(\bigcap_{n=1}^{\infty} W_n)^{\bullet}$ is $\sigma(E^*, E)$-separable. Thus by Theorem 5.2.1, we have that $U \cap V$ is a δ-ultrabarrel in (E, ξ) and so $U \cap V \in \mathcal{U}$. Hence (i) is also satisfied. It follows from the fact that (E, ξ) is δ-ultrabarrelled that the vector space topology on E with $\mathcal{U}$ as a base of neighbourhoods of the origin is coarser than ξ .

Definition 5.2.1

If (E, ξ) is a δ-ultrabarrelled space, we call the vector space topology on E for which the δ-ultrabarrels in (E, ξ) form a base of neighbourhoods of the origin the <u>δu-topology</u> on E and denote it by $\delta u(\xi)$.

We recall the following definition given in [22, §2].

Definition 5.2.2

An ultrabarrel U is said to be of type (α) if for each positive integer j, there is a sequence $(U_j^{(n)} : n = 0, 1, 2, \ldots)$ of closed balanced neighbourhoods of the origin in E such that for all $n \in \mathbb{N}$, $U_j^{(n+1)} + U_j^{(n+1)} \subseteq U_j^{(n)}$, $U^{(n)} = \bigcap_{j=1}^{\infty} U_j^{(n)}$ is absorbent and $U^{(o)} = U$. Note that $(U^{(n)})$ is a defining sequence for U. A topological vector space E is countably ultrabarrelled if every ultrabarrel of type (α) is a neighbourhood of the origin in E.

Theorem 5.2.5

Let (E, ξ) be a δ-ultrabarrelled space. Then E is both δ-ultrabarrelled and countably ultrabarrelled under $\delta u(\xi)$.

Proof

Let B be a δ-ultrabarrel in $(E, \delta u(\xi))$. Since ξ is finer than $\delta u(\xi)$, we have that B is ξ-closed and so is a δ-ultrabarrel in (E, ξ). It follows from the construction of $\delta u(\xi)$ that B is a $\delta u(\xi)$-neighbourhood of the origin. Thus E is δ-ultrabarrelled under $\delta u(\xi)$.

Let U be an ultrabarrel of type (α) in $(E, \delta u(\xi))$. Then for each $j \in \mathbb{N}$, there is a sequence $(U_j^{(n)})_{n \geq o}$ of closed balanced $\delta u(\xi)$-neighbourhoods of the origin such that

(i) $U_j^{(n+1)} + U_j^{(n+1)} \subseteq U_j^{(n)}$, $n = 0, 1, 2, \ldots$

(ii) $U^{(n)} = \bigcap_{j=1}^{\infty} U_j^{(n)}$ is absorbent for all $n = 0, 1, 2, \ldots$

(iii) $U^{(o)} = U$.

For each j, let $(V_j^{(n+1)})_{n\geq 1}$ be a defining sequence for $U_j^{(1)}$ such that if $N_j = \bigcap_{n=1}^{\infty} V_j^{(n+1)}$ then $\dim E/N_j \leq c$. Let

$W_j^{(n)} = \overline{U_j^{(n)} + N_j}$, the $\delta u(\xi)$-closure of $U_j^{(n)} + N_j$ $(n, j \in \mathbb{N})$.

Then

$$W_j^{(n+1)} + W_j^{(n+1)} \subseteq \overline{U_j^{(n+1)} + U_j^{(n+1)} + N_j} \subseteq \overline{U_j^{(n)} + N_j} = W_j^{(n)} \quad (n, j \in \mathbb{N}).$$

Each $W_j^{(n)}$ is a $\delta u(\xi)$-neighbourhood of the origin and $(W_j^{(n)})_{n\geq 2}$ is a defining sequence for $W_j^{(1)}$. Since $N_j \subseteq \bigcap_{n=2}^{\infty} W_j^{(n)}$, we have $\dim E/\bigcap_{n=2}^{\infty} W_j^{(n)} \leq \dim E/N_j \leq c$. Also, $U^{(n)} = \bigcap_{j=1}^{\infty} U_j^{(n)} \subseteq \bigcap_{j=1}^{\infty} W_j^{(n)}$ so that $\bigcap_{j=1}^{\infty} W_j^{(n)}$ is absorbing for each $n = 1, 2, \ldots$ Hence (again reasoning as in Lemma 3.2.2) we have by Theorem 5.2.1, that $\bigcap_{j=1}^{\infty} W_j^{(1)}$ is a δ-ultrabarrel for the original topology and so is a neighbourhood of the origin in $(E, \delta u(\xi))$. But we have

$$\bigcap_{j=1}^{\infty} W_j^{(1)} = \bigcap_{j=1}^{\infty} (\overline{U_j^{(1)} + N_j}) \subseteq \bigcap_{j=1}^{\infty} (\overline{U_j^{(1)} + U_j^{(1)}}) \subseteq \bigcap_{j=1}^{\infty} \overline{U_j^{(o)}} = \bigcap_{j=1}^{\infty} U_j^{(o)} = U^{(o)}.$$

Hence $U = U^{(o)}$ is a neighbourhood of the origin in $(E, \delta u(\xi))$. Thus $(E, \delta u(\xi))$ is countably ultrabarrelled.

<u>Example 5.2.1</u>

Let Λ be a set such that $|\Lambda| > c$. Consider the space $\ell_{\frac{1}{2}}(\Lambda) = \{(\xi_\lambda)_{\lambda\in\Lambda} : \xi_\lambda \in \mathbb{R}, \sum_{\lambda\in\Lambda} |\xi_\lambda|^{\frac{1}{2}} < \infty\}$ with the topology $\mathcal{J}$ defined by the metric $d((\xi_\lambda)_{\lambda\in\Lambda}, (\eta_\lambda)_{\lambda\in\Lambda}) = \sum_{\lambda\in\Lambda} |\xi_\lambda - \eta_\lambda|^{\frac{1}{2}}$.

This is a non-locally convex topology and one can show as with the ordinary $\ell_{\frac{1}{2}}$ space that $\ell_{\frac{1}{2}}(\Lambda)$ is complete under $\mathcal{J}$. Consequently $(\ell_{\frac{1}{2}}(\Lambda), \mathcal{J})$ is ultrabarrelled [44, Corollary to Proposition 12] and therefore δ-ultrabarrelled. We note further that $\dim \ell_{\frac{1}{2}}(\Lambda) > c$.

Now let $U = \{(\xi_\lambda)_{\lambda\in\Lambda} : \sum_{\lambda\in\Lambda} |\xi_\lambda|^{\frac{1}{2}} \le 1\}$. Then U is an ultrabarrel. Let (U_n) be any defining sequence for U . Then $N = \bigcap_{n=1}^{\infty} U_n = \{0\}$, , for if $x \in N$, since N is a subspace, $\alpha x \in N$ for all $\alpha \in K$. It follows that $\alpha x \in U$ for all $\alpha \in K$. Suppose $x = (\xi_\lambda)_{\lambda\in\Lambda}$. Then $1 \ge d(\alpha x, 0) = |\alpha|^{\frac{1}{2}} \sum_{\lambda\in\Lambda} |\xi_\lambda|^{\frac{1}{2}}$ for all $\alpha \in K$, which is impossible unless $\sum_{\lambda\in\Lambda} |\xi_\lambda|^{\frac{1}{2}} = 0$ in which case $\xi_\lambda = 0$ for all λ and so $x = 0$. It therefore follows that U is not a δ-ultrabarrel, since $\dim \ell_{\frac{1}{2}}(\Lambda)/_N = \dim \ell_{\frac{1}{2}}(\Lambda) > c$. Consequently $\delta u(\xi)$ is coarser than $\mathcal{J}$.

Let Ω be a subset of Λ with $|\Omega| \le \aleph_0$, let t be a positive real number and put

$$W = \{(\xi_\lambda)_{\lambda\in\Lambda} \in \ell_{\frac{1}{2}}(\Lambda) : \sum_{\lambda\in\Omega} |\xi_\lambda|^{\frac{1}{2}} \le t\} \quad . \qquad (*)$$

It is not difficult to see that W is balanced and absorbent. We show that W is also closed. Let $y = (\zeta_\lambda)_{\lambda\in\Lambda} \in \overline{W}$ and let (y_n) be a sequence in W converging to y , where $y_n = (\xi_\lambda^{(n)})_{\lambda\in\Lambda}$ $(n \in \mathbb{N})$. Then for all $\epsilon > 0$, we have

$$\begin{aligned}\sum_{\lambda\in\Omega} |\zeta_\lambda|^{\frac{1}{2}} &\le \sum_{\lambda\in\Omega} |\zeta_\lambda - \xi_\lambda^{(n)}|^{\frac{1}{2}} + \sum_{\lambda\in\Omega} |\xi_\lambda^{(n)}|^{\frac{1}{2}} \\ &\le \sum_{\lambda\in\Lambda} |\zeta_\lambda - \xi_\lambda^{(n)}|^{\frac{1}{2}} + t \\ &\le \epsilon + t \quad ,\end{aligned}$$

for all sufficiently large n . Hence $\sum_{\lambda\in\Omega} |\zeta_\lambda|^{\frac{1}{2}} \le t$ and so $y \in W$. Thus W is closed.

It is clear that if $W_n = \{(\xi_\lambda)_{\lambda\in\Lambda} \in \ell_{\frac{1}{2}}(\Lambda) : \sum_{\lambda\in\Omega} |\xi_\lambda|^{\frac{1}{2}} \le \frac{1}{2^n} t\}$ $(n \in \mathbb{N})$, then $W_1 + W_1 \subseteq W$ and $W_{n+1} + W_{n+1} \subseteq W_n$ $(n \in \mathbb{N})$. Also as above each W_n is closed balanced and absorbent so that W is an ultrabarrel. Now $\bigcap_{n=1}^{\infty} W_n = \{(\xi_\lambda)_{\lambda\in\Lambda} \in \ell_{\frac{1}{2}}(\Lambda) : \xi_\lambda = 0 \text{ if } \lambda \in \Omega\}$ and

$\dim \{(\xi_\lambda)_{\lambda\in\Lambda} \in \ell_{\frac{1}{2}}(\Lambda) : \xi_\lambda = 0 \text{ if } \lambda \notin \Omega\} \le c$. Consequently $\dim E/\bigcap_{n=1}^{\infty} W_n \le c$ so that W is a δ-ultrabarrel.

Now let $x = (\eta_\lambda)_{\lambda\in\Lambda}$ be a non-zero element of $\ell_{\frac{1}{2}}(\Lambda)$. Then there exists a positive real number r such that $x \notin \{(\xi_\lambda)_{\lambda\in\Lambda} \in \ell_{\frac{1}{2}}(\Lambda) : \sum_{\lambda\in\Lambda} |\xi_\lambda|^{\frac{1}{2}} \le r\}$. If we construct W as above with $t = r$ and $\Omega = \{\lambda\in\Lambda : \eta_\lambda \ne 0\}$, then W is a $\delta u(\xi)$-neighbourhood of 0 which does not contain x . This topology $\delta u(\xi)$ is therefore separated.

According to S.O. Iyahen [24, §3], a semiconvex space E is said to be hyperbarrelled if every closed balanced semiconvex absorbent subset of E is a neighbourhood of the origin in E . A semiconvex space E is hyperbarrelled if and only if every linear mapping with a closed graph of E into any complete separated locally bounded space is continuous [24, Theorem 3.3].

In a way similar to Definition 5.2.1, we define a δ-hyperbarrelled space F as a semiconvex space in which every closed balanced semiconvex absorbent subset V such that $\dim F/\bigcap_{\lambda>0}\lambda V \le c$ is a neighbourhood of the origin. Similar techniques give results corresponding to some of the results in the case of the δ-ultrabarrelled spaces. In particular, we get that a semiconvex space E is δ-hyperbarrelled if and only if whenever $t : E \to F$ is a linear mapping with a closed graph of E into a complete locally bounded topological vector space of dimension at most c , t is continuous.

5.3 Generalizations to Arbitrary Infinite Cardinals

In contrast to the finite dimensional situation, linear dimension does not generally play a significant role in the study of topological vector spaces. It does appear in our previous considerations of essentially separable sets and δ-barrelled spaces, but there it obscures an underlying general principle from which we may generalize the idea of a δ-barrelled space.

Let E be a locally convex space, let B be a barrel in E and let α be an infinite cardinal number. If $q : E \to E/\bigcap_{\lambda>0}\lambda B$ is the quotient map, the Minkowski functional of $q(B)$ is a norm on $E/\bigcap_{\lambda>0}\lambda B$. If $E/\bigcap_{\lambda>0}\lambda B$ has a dense subset of cardinality at most α for the resulting norm topology, we say that B is a <u>$\delta(\alpha)$-barrel</u>. A locally convex space in which each $\delta(\alpha)$-barrel is a neighbourhood of the origin will be called a <u>$\delta(\alpha)$-barrelled space</u>.

We know that a normed space with a dense subset of cardinality at most c has dimension at most c (Lemma 2.1.4). The converse is trivial so that $\delta(c)$-barrels and δ-barrels are the same and consequently the classes of $\delta(c)$-barrelled spaces and δ-barrelled spaces coincide.

The following result extends Theorem 3.1.2. We recall that the density character of a Banach space is the smallest cardinal for a dense subset of the space [32].

<u>Theorem 5.3.1</u>

A locally convex space E is $\delta(\alpha)$-barrelled if and only if whenever F is a Banach space of density character at most α and $t : E \to F$ is a linear mapping with a closed graph, t is continuous.

<u>Proof</u>

We refer to the proof of Theorem 3.1.2 and indicate necessary modifications, retaining the same notations.

Suppose $\{y_\lambda : \lambda \in \Lambda\}$ is a dense subset of F with $|\Lambda| \leq \alpha$. Choose an element in each non-empty $\{y \in F : \|y - y_\lambda\| \leq \frac{1}{n}\} \cap t(E)$, $(n \in \mathbb{N}, \lambda \in \Lambda)$. Clearly the set D of all these elements has cardinality at most α. We show that D is dense in $t(E)$. Given $z \in t(E)$ and $\epsilon > 0$, choose $n \in \mathbb{N}$, y_λ and $y \in D$ such that $\frac{1}{n} \leq \frac{\epsilon}{2}$, $\|z - y_\lambda\| \leq \frac{1}{n}$ and $\|y - y_\lambda\| \leq \frac{1}{n}$. Then $\|z - y\| \leq \frac{2}{n} \leq \epsilon$.

Let $D = \{z_\mu : \mu \in M\}$ $(|M| \leq \alpha)$. For each z_μ, choose $x_\mu \in E$ such that $t(x_\mu) = z_\mu$. Given $x \in E$ and $\varepsilon > 0$, we can choose $\mu_o \in M$ such that $t(x) - z_{\mu_o} \in \varepsilon B$, where B is the closed unit ball of F. Then $x - x_{\mu o} \in t^{-1}(\varepsilon B) \subseteq \varepsilon \overline{t^{-1}(B)}$. It now follows that if $N = \bigcap_{\lambda > 0} \lambda \overline{t^{-1}(B)}$ then the set of equivalence classes in $E/_N$ of the elements x_μ $(\mu \in M)$ is dense for the norm defined as above from $\overline{t^{-1}(B)}$ which is therefore a $\delta(\alpha)$-barrel.

The proof of the necessity of the condition is concluded as in Theorem 3.1.2. The only modification required in the sufficiency part is to note that if a normed space has a dense subset of cardinality α, then so has its completion.

To obtain an analogue of Theorem 3.1.1 we have to generalize the characterization of bounded absolutely convex essentially separable sets given in Theorem 2.1.7.

<u>Theorem 5.3.2</u>

A separated locally convex space E is $\delta(\alpha)$-barrelled if and only if each $\sigma(E', E)$-bounded absolutely convex set A such that $\sigma(E', E)|_A$ has a base of neighbourhoods of 0 consisting of at most α sets is equicontinuous.

Proof

Suppose that A is a non-empty set with the properties stated in the Theorem. Then A° is a barrel in E and the normed space constructed as above is just $\mathcal{N}(E, A)$ (section 2.1). Arguing as in Theorem 2.1.7, we deduce that $\mathcal{N}(E, A)$ has a total subset T of cardinality at most α. The set D of linear combinations with rational coefficients in the real case, complex rational coefficients in the complex case, of elements of T is dense in $\mathcal{N}(E, A)$. Since α is infinite, we also have $|D| \leq \alpha$, so that A° is a $\delta(\alpha)$-barrel. Thus if E is $\delta(\alpha)$-barrelled, A is equicontinuous as required.

Suppose that B is a $\delta(\alpha)$-barrel in E. Let D be a dense subset of $\mathcal{N}(E, B^{\circ})$ with cardinality at most α. Since $B^{\bullet}$ is $\sigma(E^*, E)$-compact, the coarsest topology on $B^{\bullet}$ which makes each of the elements of D continuous must coincide with $\sigma(E^*,E)|_{B^{\bullet}}$. It now follows easily that B° has the properties enumerated for A. Thus if the condition is satisfied, B° is equicontinuous so that $B = B^{\circ\circ}$ is a neighbourhood of zero, i.e. E is $\delta(\alpha)$-barrelled.

Since the scalar field has a countable base for its topology, we deduce from the second part of the proof of the previous theorem

Corollary

Let A be a $\sigma(E', E)$-bounded absolutely convex set such that $\sigma(E', E)|_A$ has a base of neighbourhoods of 0 consisting of at most α sets where α is an infinite cardinal. Then $\sigma(E', E)|_A$ has a base consisting of at most α sets and A has a $\sigma(E', E)$-dense subset of cardinality at most α.

When $\alpha = \aleph_0$, the condition of the last theorem on the $\sigma(E', E)$-bounded absolutely convex set A is equivalent to its metrizability under $\sigma(E', E)|_A$. This is established in Proposition 1.3 of [25]. We then have from the theorem and Theorem 2.6 of [25] that the class of separated $\delta(\aleph_0)$-barrelled spaces coincides with Kalton's class $\mathcal{B}(\zeta_B)$.

It is immediate from the definition that if α, β are infinite cardinals such that $\alpha < \beta$, then each $\delta(\beta)$-barrelled space is also $\delta(\alpha)$-barrelled. However we can always find $\delta(\alpha)$-barrelled spaces which are not $\delta(\beta)$-barrelled. Take for example $E = \mathbb{R}^{(M)}$ where $|M| = \beta$ and $E' = \{(\xi_\mu)_{\mu \in M} \in \mathbb{R}^M : |\{\mu : \xi_\mu \neq 0\}| \leq \alpha\}$. Using the Corollary to Theorem 5.3.2 and arguing as in Examples 2.2.1 and 3.1.1(a), we see that $(E, \tau(E, E'))$ provides such an example.

Note that in contrast to the situation with the dimension of a complete metrizable topological vector space if we reject the continuum hypothesis and assume that there is a cardinal γ such that $\aleph_0 < \gamma < c$, then there is a $\delta(\gamma)$-barrelled space which is not δ-barrelled and consequently there is a Banach space of dimension c whose density character is strictly greater than γ.

We note finally that $\delta(\alpha)$-barrelled spaces have the countable codimension property. The proof of the result has much in common with that of Theorem 3.1.6. However the approach in the proof of Theorem 3.1.6 makes use of certain basic properties of essentially separable sets which do not appear to generalize.

<u>Theorem 5.3.3</u>

Let E be a $\delta(\alpha)$-barrelled space and let X be a subspace of countable codimension. Then X is $\delta(\alpha)$-barrelled in the topology induced from E.

Proof

We consider first the separated case. Let A be a non-empty $\sigma(X', X)$-bounded absolutely convex set such that $\sigma(X', X)|_A$ has a base of neighbourhoods of 0 consisting of at most α sets. Keeping the same notations as in the proof of Theorem 3.1.6, we construct B and C as before. The $\sigma(E', E)$-sequential completeness of E' comes from [25, Corollary to Theorem 1.4 and Theorem 2.6] and the fact that E is also $\delta(\mathfrak{H}_o)$-barrelled.

There is a family $(\phi_\lambda)_{\lambda\in\Lambda}$ of non-empty finite subsets of X such that $|\Lambda| \leq \alpha$ and the sets $\{x' \in A : |\langle x, x'\rangle| \leq 1 , x \in \phi_\lambda\}$ $(\lambda\in\Lambda)$ form a base of neighbourhoods of zero for $\sigma(X', X)|_A$. We may assume that $G \neq X$, $G \neq E$ (otherwise simple modifications give the result). Let $\{y_n : n \in \mathbb{N}\}$ be an at most countable subset of $G\backslash X$ which spans a supplement of X in G and let $\{x_n : n \in \mathbb{N}\}$ be an at most countable subset of $E\backslash G$ which spans a supplement of G in E. We show that the family Υ of sets

$$\{x' \in C : |\langle x, x'\rangle| \leq 1 , x \in \phi_\lambda\} \cap \{x' \in C : |\langle y_s , x'\rangle| \leq \tfrac{1}{M} , s = 1, \ldots, M\}$$

$(\lambda \in \Lambda , M \in \mathbb{N})$ is a base of neighbourhoods of zero for $\sigma(E', E)|_C$.

Let U be any neighbourhood of 0 for $\sigma(E', E)|_C$ and choose $\omega_1, \ldots, \omega_n \in E$ such that

$$\{x' \in C : |\langle\omega_r, x'\rangle| \leq 1 , r = 1, \ldots, n\} \subseteq U .$$

Let $\omega_r = z_r + \sum\limits_{s\in\theta_r} \alpha_s^{(r)} y_s + \sum\limits_{t\in\theta'_r} \beta_t^{(r)} x_t$, where θ_r, θ'_r are non-empty finite subsets of $\mathbb{N}$ and $z_r \in X$ $(r = 1, \ldots, n)$.

Let $R = \max \{|\alpha| : \alpha \in \bigcup\limits_{r=1}^{n} \{\alpha_s^{(r)} : s \in \theta_r\}\}$

$N = \max \{s : s \in \bigcup\limits_{r=1}^{n} \theta_r\}$.

Choose λ_o such that

$$\{x' \in A : |\langle x, x'\rangle| \leq 1, x \in \phi_{\lambda_o}\} \subseteq \{x' \in A : |\langle z_r, x'\rangle| \leq \tfrac{1}{2} , r = 1, \ldots, n\} .$$

We have that

$$V = \{x' \in C : |\langle x, x'\rangle| \leq 1,\ x \in \phi_{\lambda_0}\} \cap \{x' \in C : |\langle y_s, x'\rangle| \leq \frac{1}{M},$$

$$s = 1, \ldots, M\} \subseteq U, \quad \text{where } M \in \mathbb{N} \text{ and } M \geq 2N(1 + R),$$

for if $x' \in V$, then we have for each $r \in \{1, \ldots, n\}$,

$$|\langle \omega_r, x'\rangle| \leq |\langle z_r, x'|_X\rangle| + \sum_{s \in \theta_r} |\alpha_s^{(r)}|\, |\langle y_s, x'\rangle| + \sum_{t \in \theta'_r} |\beta_t^{(r)}|\, |\langle x_t, x'\rangle|$$

so that $|\langle \omega_r, x'\rangle| \leq \frac{1}{2} + N.\ R.\ \frac{1}{M} + 0 \leq \frac{1}{2} + \frac{1}{2} = 1$.

Thus $\mathfrak{T}$ is a base of neighbourhoods of 0 for $\sigma(E', E)|_C$. Since $|\Lambda| \leq \alpha$ and α is infinite, it follows easily that $|\mathfrak{T}| \leq \alpha$ and since E is $\delta(\alpha)$-barrelled we have by Theorem 5.3.2 that C is equicontinuous. Consequently as in the proof of Theorem 3.1.6, it follows that A is equicontinuous. Hence X is $\delta(\alpha)$-barrelled as required.

If E is not separated, we use the analogue of Theorem 3.1.3 together with the first part to obtain the result (c.f. proof of Theorem 3.1.6).

The idea of essential separability arose from our initial attempts to characterize σ-barrelled spaces by means of a closed graph theorem. We have in fact a general result characterizing the γ-barrelled spaces of Valdivia mentioned in Example 3.2.1.

<u>Theorem 5.3.4</u>

Let (E, ξ) be a separated locally convex space and let γ be an infinite cardinal. The following are equivalent:

(i) E is γ-barrelled;

(ii) whenever $t : E \to F$ is a linear mapping such that

(a) F is a Banach space which is the dual of a Banach space G with density character at most γ,

(b) the graph of t is closed in $(E, \xi) \times (F, \sigma(F, G))$,

then t is continuous under ξ and the norm topology of F ;

(iii) same assertion as (ii) with $F = \ell_\infty(\Lambda)$, $G = \ell_1(\Lambda)$ where $|\Lambda| = \gamma$.

<u>Proof</u>

<u>(i) $\Longrightarrow$ (ii)</u> Suppose that E is γ-barrelled and let $t : E \to F$ be as in (ii). Consider the transpose $t' : F' \to E^*$ of t . By the hypothesis, $t'^{-1}(E')$ contains a $\sigma(G, F)$-dense vector subspace of G . Since such a vector subspace is also dense in G under its norm topology $\tau(G, F)$, each element y' of G is the limit under $\tau(G, F)$ of a sequence (y'_n) in $t'^{-1}(E') \cap G$. Since E is also σ-barrelled, $\{t'(y'_n) : n \in \mathbb{N}\}$ is an equicontinuous subset of E' and so its $\sigma(E^*, E)$-closure is contained in E' . This implies that $t'(y') \in E'$ and consequently $G \subseteq t'^{-1}(E')$.

The closed unit ball B of G is dense in the closed unit ball B' of F' under $\sigma(F', F)$. If X is any dense subset of cardinality at most γ in the Banach space G , then $X \cap B$ is a $\sigma(F', F)$-dense subset of B' of cardinality at most γ . Now $t'(X \cap B)$ is $\sigma(E', E)$-bounded and therefore equicontinuous since E is γ-barrelled. It follows as before that $t'(B') \subseteq E'$ and is equicontinuous. Since B' spans F' it follows that $t'(F') \subseteq E'$ and that t is continuous as required.

<u>(ii) $\Longrightarrow$ (iii)</u> In $\ell_1(\Lambda)$ let $e_\mu = (\delta_{\lambda\mu})_{\lambda\in\Lambda}$ $(\mu \in \Lambda)$. We obtain a dense subset of $\ell_1(\Lambda)$ of cardinality γ by taking the set of all elements of the form $\sum_{r=1}^{n} \alpha_r e_{\mu(r)}$, where n is a positive

integer and the α_r are rational in the real case and complex rational in the complex case. Thus $\ell_1(\Lambda)$ has density character (at most) γ. It then follows that (iii) is a special case of (ii).

<u>(iii) $\Longrightarrow$ (i)</u> Let $(x_\lambda)_{\lambda\in\Lambda}$ be a $\sigma(E', E)$-bounded family with $|\Lambda| \le \gamma$. We may assume without loss of generality that $|\Lambda| = \gamma$ since otherwise we may extend the family by introducing sufficiently many x'_α, all equal to zero. Consider the mapping $t : E \to \ell_\infty(\Lambda)$ defined by $t(x) = (\langle x, x'_\lambda\rangle)_{\lambda\in\Lambda}$. Clearly t is linear and if $(\alpha_\lambda)_{\lambda\in\Lambda}$ is any element of $\ell_1(\Lambda)$ with only finitely many non-zero components and if $t' : \ell_\infty(\Lambda)' \to E^*$ is the transpose of t, then for each $x \in E$, we have

$$\langle x, t'((\alpha_\lambda)_{\lambda\in\Lambda})\rangle = \sum_{\lambda\in\Lambda} \alpha_\lambda \langle x, x'_\lambda\rangle = \langle x, \sum_{\lambda\in\Lambda} \alpha_\lambda x'_\lambda\rangle \quad .$$

Thus $t'((\alpha_\lambda)_{\lambda\in\Lambda}) = \sum_{\lambda\in\Lambda} \alpha_\lambda x'_\lambda$ and so

$t'^{-1}(E') \supseteq \{(\alpha_\lambda)_{\lambda\in\Lambda} : |\{\lambda : \alpha_\lambda \neq 0\}| < \aleph_0\}$, which is a $\sigma(\ell_\infty(\Lambda)', \ell_\infty(\Lambda))$-dense vector subspace of $\ell_1(\Lambda)$. This shows that the graph of t is closed for $\xi \times \sigma(\ell_\infty(\Lambda), \ell_1(\Lambda))$.

It follows by the hypothesis that t is continuous under ξ and the norm topology of $\ell_\infty(\Lambda)$. If B is the closed unit ball of $\ell_\infty(\Lambda)'$, we then have that $t'(B)$ is an equicontinuous subset of E'. But $A = \{(\delta_{\lambda\mu})_{\lambda\in\Lambda} : \mu \in \Lambda\} \subseteq B$ and $t'(A) = \{x'_\lambda : \lambda \in \Lambda\}$. Thus $\{x'_\lambda : \lambda \in \Lambda\}$ is an equicontinuous subset of E' and so E is γ-barrelled.

<u>Remark</u>

We observe that the σ-barrelled spaces correspond to the case $\gamma = \aleph_0$. The Banach spaces F in (ii) are then the dual spaces of separable Banach spaces and the spaces of (iii) are the usual ℓ_1 and ℓ_∞. In the case $\gamma = c$, the Banach spaces F in (ii) are the duals of Banach spaces with dimension at most c.

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