COLLECTIVE CHOICE : A PROBABILISTIC ANALYSIS

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ABSTRACT

A group of N members wishing to select one of m alternative courses of action, A_1, A_2, ..., A_m, may reach a decision either with the aid of a collective choice rule or through the informal operation of an implicit social decision scheme. Under investigation are three collective choice rules, viz. the plurality, Condorcet, and Borda procedures, and seven social decision scheme models, viz. majority, proportionality, equiprobability, highest expected value, majority if A_g but proportionality otherwise, majority with proportionality, and majority with equiprobability.

A probabilistic choice model is adopted which provides the likelihood of a given distribution of the group members over the possible preference orderings of the alternatives. By means of the model, expressions are derived, in each of the ten collective choice procedures, for the probability that A_i, i = 1, 2, ..., m, is selected by the group. Additionally, in the case of the Condorcet procedure, a recursion relation is developed which expresses the probability that A_i is the winning alternative when the group consists of N members in terms of the probability of the same event when the group contains N - 1 members. These results form the basis of a study encompassing both normative and descriptive aspects of social choice.

The examination of collective choice rules, which is primarily normative in character, concentrates on two central issues. Firstly, it is proposed that decisiveness, i.e. the tendency to yield unambiguous, clear - cut outcomes, is a desirable property of a collective choice rule, and hence

may be adopted as one of the criteria in terms of which rival social choice functions may be evaluated. To this end, expressions are developed for the likelihood of plurality, Condorcet, and Borda indecision, and a comparative analysis of these likelihoods is undertaken. Secondly, collective choice rules may produce inconsistent or anomalous outcomes. The plurality and Borda procedures may select an alternative other than the one preferred by a majority of the group members; the Condorcet procedure may give rise to the paradox of voting; and the Borda procedure may generate the reversed - order paradox. In order to gauge the seriousness of the problem facing a given collective choice rule, solutions are obtained for the likelihood of each type of anomalous result.

In the analysis of social decision scheme models descriptive considerations predominate. The implications of each model are studied, and an experimental approach is suggested which provides effective discrimination between competing models.

C O N T E N T S

CHAPTER 1

INTRODUCTION

1.1 APPROACH AND AIMS : A GENERAL OUTLINE

Groups of all kinds are frequently required to choose one out of a number of alternative courses of action. For example, an appointments committee has responsibility for selecting a suitable person for a post from a pool of applicants; several friends arranging to meet in town for an evening meal have to decide which eating establishment they will patronise; voters in a general election choose from the candidates listed on the ballot paper the one which they wish to return as their member of parliament. In each case, individual preferences are combined in some manner to form a social choice. The present study focuses on some of the methods by which the diverse preferences of group members are translated into collective action.

This inquiry differs from previous treatises in the area (e.g. Luce and Raiffa, 1957) in that the rewards and costs of decision making, whether for the individual or for the group, are not considered directly. That is, game theoretical notions of utility maximisation do not play any part in the present analysis. Of primary concern is the influence of structural factors on collective choice, namely, the number of alternatives, the number of group members, and the collective choice procedure. The influence of content (e.g. the nature of the alternatives) on collective choice is viewed in terms of the operation of a shared value system which governs the probability that an individual holds a particular preference ordering of the alternatives.

Research in the field of collective choice is generally conducted from either a normative or a descriptive standpoint. The emphasis of normative work is on the search for the method of amalgamating preferences which best fulfills certain criteria of fairness, reasonableness, and representativeness. Descriptive studies, on the other hand, investigate the procedures actually employed by groups when merging preferences. Whereas normative research is concerned with explicitly formalised collective choice rules, descriptive work deals mainly with informal procedures which function on the basis of a tacit understanding among group members.

With the notable exceptions of game theory (e.g. Luce and Raiffa, 1957) and studies of the reliability of group decisions (e.g. Smoke and Zajonc, 1962), normative issues in collective choice have generally been eschewed by psychologists who, for the most part, choose to regard them as the province of the economist and the political scientist. This is particularly unfortunate in that the work of psychologists frequently involves the aggregation of data concerning the preferences, opinions, feelings, or perceptions of a group of subjects in relation to a number of alternatives. For example, a clinical psychologist might ask a sample of agoraphobics to contemplate a number of exposed situations in order to establish whether any one situation is more likely than the others to generate apprehension among agoraphobics; or, in a social psychology experiment, observers examining the behaviour of a group might be required to indicate the member whom they consider best fits the description of leader. Now, the problem of assessing the merits of an index of group opinion is similar to the problem of evaluating the desirability of a collective choice method. In this respect, the goals of the psychologist and the social choice theorist overlap : both wish to arrive at a method which will faithfully reflect the views held by a group of individuals. Thus, research into normative aspects of social choice has relevance for questions of psychological measurement (cf. Coombs, 1964, chapter 18).

As might be expected, psychologists have pursued the descriptive approach with greater zeal, so that there is a substantial body of research on this aspect of collective choice (e.g. Davis, 1969, 1973; Fisher, 1974; Lieberman, 1971; Steiner, 1972). Recent empirical findings in the area are reviewed by Davis (1976). Surprisingly, however, very few models of the group process underlying social choice behaviour have been cast in a mathematical form. A perusal of the major texts and review articles in the field of mathematical social psychology (e.g. Abelson, 1967; Berger et al, 1962; Coleman, 1960; Criswell et al, 1962; Rapoport, 1963; Rosenberg, 1968) reveals a distinct absence of quantitative models of collective choice behaviour outside the game theoretical tradition, apart from the Voting model of Hays and Bush (1954) and the two models of group problem - solving behaviour formulated by Lorge and Solomon (1955). All three models are somewhat specialised, dealing only with the case of two alternatives and, moreover, alternatives of the "correct/incorrect" type. Recently, however, Davis (1973) has more than filled the void with his general social decision scheme model from which a plethora of interesting special cases may be derived. Unfortunately, in most of these special cases the predictions of the model, which take the form of probabilities for each social choice outcome, have so far had to be obtained by means of a computer enumeration routine since explicit, closed - form expressions for the probabilities have as yet not been achieved.

The present study examines both normative and descriptive aspects of collective choice, with an emphasis on the former. The connecting link between the two approaches is supplied by a probabilistic choice model which yields the likelihood of a given distribution of group members over the possible preference orderings of the alternatives. (In so far as Davis's (1973) general social decision scheme model considers first preferences only, then in this respect it constitutes a special case of the present model.) With the assistance of the model, three main subjects of inquiry are pursued. Firstly, the likelihood of collective indecision is investigated. That is, for each of the formal collective choice systems under consideration, an expression is derived for the probability that an indecisive, or equivocal, outcome will arise. Secondly, these collective choice rules, which may be regarded as analogues of psychological data analysis procedures, can also give rise to various inconsistent or paradoxical outcomes. An analysis is performed in each case in order to arrive at a solution for the likelihood of each anomalous result. Thirdly, altogether under investigation are three formal methods of collective choice and seven social decision scheme models. Expressions for the likelihood that a given alternative is selected by the group are developed in all ten instances. Since the solutions in the case of the formal procedures have "considerable potential for the description/prediction of empirical data as well" (Davis, 1976, p. 515), we effectively achieve explicit, closed - form probability expressions for ten social decision scheme models.

The next two sections in this chapter examine in more detail the background and aims of the normative part of the present study.

1.2 COLLECTIVE CHOICE RULES

Three formal methods of collective choice are studied, namely, the plurality, Condorcet, and Borda procedures. The most widely used of the three is the plurality procedure which plays a prominent role in the electoral systems of Great Britain, the United States, and Canada (Rae, 1967). The Condorcet and Borda procedures are generally regarded by social choice theorists (e.g. Black, 1958; Fishburn, 1973) as superior to the plurality method. However, the philosophies embodied by the

Condorcet and Borda systems are quite different from each other. Before describing the procedures some preliminary notation will be helpful.

Let N denote the number of group members, or voters ; let m denote the number of alternatives ; and let A_j denote the j^{th} alternative. The vertical arrangement

$$\begin{array}{c} A_e \\ A_f \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ A_g \end{array}$$

represents a linear preference ordering of the m alternatives in which each alternative is preferred to every other alternative which lies below it.

Plurality procedure. Under this system, only the first preferences of the group members are considered. The collective choice is the alternative which receives the largest number of first preference votes. For example, when m = 3 and N = 15, if the voters are distributed over the preference orderings in the following manner

Example 1.1	A_1	A_1	A_2	A_2	A_3	A_3
m = 3 ; N = 15	A_2	A_3	A_1	A_3	A_1	A_2
	A_3	A_2	A_3	A_1	A_2	A_1
	3	5	2	2	1	2

then, of the first preference votes, A_1 receives 8, A_2 receives

4, and A_3 receives 3. Hence, A_1 is deemed the plurality winner. Although each group member's full preference ordering is recorded in Example 1.1, the same result could have been achieved more economically by the more common practice of recording each voter's first preference only. It is in the simplicity of its operation that the appeal of the plurality procedure largely resides.

Condorcet procedure. All pairwise combinations of the alternatives are considered. For each pair, by counting the number of voters' preference orderings in which one alternative is ranked higher than the other, a simple majority winner is obtained. The Condorcet winner is the alternative, if one exists, which defeats all the others in pairwise simple majority contests. For example, when m = 3 and N = 21, consider the following preference configuration of the group members :

Example 1.2	A_1	A_1	A_2	A_2	A_3	A_3
m = 3; N = 21	A_2	A_3	A_1	A_3	A_1	A_2
	A_3	A_2	A_3	A_1	A_2	A_1
	3	1	4	5	2	6

Comparing A_1 with A_2 : A_1 is preferred to A_2 by 3 + 1 + 2 = 6 voters while A_2 is preferred to A_1 by 4 + 5 + 6 = 15 voters. Comparing A_1 with A_3 : A_1 is preferred to A_3 by 3 + 1 + 4 = 8 voters while A_3 is preferred to A_1 by 5 + 2 + 6 = 13 voters. Lastly, comparing A_2 with A_3 : A_2 is preferred to A_3 by 3 + 4 + 5 = 12 voters while A_3 is preferred to A_2 by 1 + 2 + 6 = 9 voters. Since A_2 defeats A_1 and A_3 in simple majority contests, by 15 votes to 6 and by 12 votes to 9 respectively, A_2 is deemed the Condorcet winner.

Borda procedure. Each voter's preference ordering of the m alternatives is dealt with in the following manner : the most preferred alternative in a preference ordering is assigned a mark of m - 1, the second most preferred a mark of m - 2, and so on down to the least preferred which is assigned a mark of 0. All marks received by an alternative from the voters are summed to form a Borda score. The alternative with the highest Borda score is declared the winner. For example, when m = 3 and N = 15, consider the following preference configuration of the group members :

Example 1.3	A_1	A_1	A_2	A_2	A_3	A_3
m = 3; N = 15	A_2	A_3	A_1	A_3	A_1	A_2
	A_3	A_2	A_3	A_1	A_2	A_1
	2	1	0	5	3	4

Now, A_1 receives a Borda score of 2(2 + 1) + (0 + 3) = 9 ; A_2 receives a Borda score of 2(0 + 5) + (2 + 4) = 16 ; and A_3 receives a Borda score of 2(3 + 4) + (1 + 5) = 20. With the highest Borda score of 20, alternative A_3 is designated the Borda winner.

A collective choice rule, such as the plurality, Condorcet, or Borda procedure, is commonly referred to as a "social choice function" (Fishburn, 1973; Richelson, 1975) when it is used to select a single winning alternative (or, in the case of a tie, more than one winning alternative), and as a "social welfare function" (Arrow, 1963; Sen, 1970) when it is used to obtain a social ordering over all the alternatives. Although the present analysis is mainly concerned with collective choice rules in their capacity as social choice functions, we shall also have occasion to consider them in the role of social welfare functions.

Traditional democratic doctrine holds that a social choice function should select the alternative preferred by a majority of the group. However, the immediate appeal of this seemingly straightforward requirement conceals a number of underlying difficulties. Consider the Condorcet method, which is essentially an embodiment of the democratic principle that the will of the majority should prevail. The following example demonstrates that a rather anomalous result can occur when the Condorcet procedure is employed.

<u>Example 1.4</u>	A_1	A_1	A_2	A_2	A_3	A_3
m = 3; N = 20	A_2	A_3	A_1	A_3	A_1	A_2
	A_3	A_2	A_3	A_1	A_2	A_1
	6	2	1	5	5	1

In pairwise simple majority contests, A_1 defeats A_2 by 13 votes to 7, A_2 defeats A_3 by 12 votes to 8, and A_3 defeats A_1 by 11 votes to 9. Thus, there is intransitivity in the social ordering, in the form of a cyclical majority, and no Condorcet winner emerges. Consequently, no alternative can be said to represent the will of the majority. The cyclical majority problem, known also as the paradox of voting, has generated a substantial body of research concerning, for example, the conditions for its occurrence (e.g. Black, 1958; Sen, 1970) and the likelihood of its occurrence (e.g. Gehrlein and Fishburn, 1976; Gillett, 1976, 1977, 1978; Niemi and Weisberg, 1972). The paradox of voting is not avoided by the expedient, commonly adopted by committees and parliamentary bodies, of voting on the alternatives two at a time. Because defeated motions are not reintroduced, such a practice merely disguises the extent of the problem (Black, 1958; Niemi and Weisberg, 1972). Moreover, the practice is open to exploitation in that "the later any motion enters the voting, the greater its chance of adoption" (Black, 1958, p. 40).

The plurality and Borda procedures do not give rise to cyclical majorities, because individual preferences are combined in a manner that does not require a true majority in order to produce a collective decision. Thus, in Example 1.4, application of either the plurality or Borda rule results in the emergence of a clear - cut winner, A_1 in both cases. On the other hand, the plurality and Borda procedures suffer from the disadvantage that they sometimes select an alternative to which at least one other alternative is preferred by a majority of the voters.

The circumstance in which there exists an alternative which a majority of group members prefers to the plurality winner has beer studied by, amongst others, Black (1958), Blyth (1972), Borda (1781), Colman and Pountney (1978), Condorcet (1785), Fishburn (1974a), Fishburn and Gehrlein (1976), Paris (1975). To see how such a situation might arise, consider the following example :

<u>Example 1.5</u>	A_1	A_1	A_2	A_2	A_3	A_3
m = 3 ; N = 15	A_2	A_3	A_1	A_3	A_1	A_2
	A_3	A_2	A_3	A_1	A_2	A_1
	4	2	0	5	0	4

The plurality winner is A_1 with 6 first preference votes. However, if each voter's full preference ordering is taken into account, then Example 1.5 illustrates two related ways in which the plurality procedure can violate the democratic ethic. Firstly, it will be observed that both A_2 and A_3 are preferred in a simple majority sense to A_1, by 9 votes to 6 in each case. In other words, the plurality winner is actually the least - preferred alternative. Blyth (1972) terms such an occurrence the "pairwise - worst - best paradox". Secondly, applying the Condorcet procedure, we find that A_2 is the Condorcet winner, being preferred to A_1 and to A_3 by 9 votes to 6 in each case. By selecting A_1 the plurality method is thwarting the manifest desires of the majority. Paris (1975) describes such an event, in which the plurality winner does not correspond with the Condorcet winner, as "plurality distortion".

The situation where there exists an alternative which a majority of group members prefers to the Borda winner has received investigation by, amongst others, Black (1958), Condorcet (1785), and Fishburn (1973, 1974b). An example of such an eventuality is

Example 1.6	A_1	A_1	A_2	A_2	A_3	A_3
m = 3 ; N = 16	A_2	A_3	A_1	A_3	A_1	A_2
	A_3	A_2	A_3	A_1	A_2	A_1
	1	5	7	1	1	1

The Borda scores of A_1, A_2, and A_3 are 20, 18, and 10, respectively. Hence, A_1 is declared the Borda winner. However, applying the Condorcet procedure, we find that A_2 is the Condorcet winner, being preferred in a simple majority sense to A_1 and to A_3 by 9 votes to 7 in each case. Thus, like the plurality procedure, the Borda method can frustrate the wishes of the majority.

The Borda method can generate another kind of inconsistent outcome, which we shall refer to as the Borda reversed - order paradox. Suppose that Borda scores are calculated in the customary manner for a set of m alternatives. An alternative is then removed from the set and the Borda scores of the remaining m - 1 alternatives are recalculated. In the reversed - order paradox it is found that, for at least one pair of alternatives, the alternative with the larger original Borda score now has the smaller revised Borda score. Consider the following example :

<u>Example 1.7</u>	A_1	A_1	A_2	A_2	A_3	A_3
m = 3 ; N = 20	A_2	A_3	A_1	A_3	A_1	A_2
	A_3	A_2	A_3	A_1	A_2	A_1
	0	8	7	0	0	5

The Borda scores for A_1, A_2, and A_3 are 23, 19, and 18, respectively. If alternative A_3 is removed then the preference orderings reduced to

A_1	A_1	A_2	A_2	A_1	A_2
A_2	A_2	A_1	A_1	A_2	A_1
0	8	7	0	0	5

or simply

A_1	A_2
A_2	A_1
8	12

The revised Borda scores for A_1 and A_2 are 8 and 12, respectively. Thus, whereas A_1 precedes A_2 in the original social ordering, the removal of A_3 results in A_2 now preceding A_1. To make the example slightly more concrete, suppose that A_3 is removed because it is discovered to be an impracticable course of action. Do we now base our choice between A_1 and A_2 on the original or on the revised Borda scores? Unfortunately, it is not at all clear how we should proceed. Our confidence in the stability of the Borda outcome is shaken. Special cases of the reversed - order paradox have been studied by Davidson and Odeh (1972) and by Fishburn (1974b).

So far we have uncovered a number of flaws in the plurality, Condorcet, and Borda procedures : the Condorcet method can result in the paradox of voting, the plurality and Borda methods can select an alternative other than the one preferred by a majority of the voters, and the Borda method can in addition give rise to the reversed - order paradox. In the light of these discoveries it is natural to inquire whether a collective choice rule exists which is immune to such difficulties. More precisely, the strategy may be adopted of investigating the conditions which a desirable collective choice procedure should fulfill, in order to determine the procedure which best meets these requirements. A second strategy, which may be pursued simultaneously, is to obtain a measure of the seriousness of the problem besetting a particular collective choice rule. If an anomaly or paradox turns out to be an exceedingly rare event, then for all practical purposes it may be disregarded. As both strategies are adopted in the present work, they will be discussed in turn under the respective headings of the conditional approach and the likelihood approach.

Conditional approach. So far our criterion for a satisfactory collective choice method has been the majority rule requirement. However, simpler more fundamental conditions may be formulated which a desirable collective choice rule should fulfill (Arrow, 1963; Fishburn, 1973; Richelson, 1975; Sen, 1970). In a classic treatise on the subject, Arrow (1963) proposed four conditions which a collective choice rule should satisfy in order to qualify as a social welfare function. (A collective choice rule, it will be recalled, is a social welfare function when used to provide a social ordering of the alternatives, and a social choice function when used to select a single winning alternative.) The conditions, which were deliberately chosen to be mild and unexceptionable, that is, necessary rather than sufficient, are

(i) unrestricted domain : the collective choice rule should operate successfully on any logically possible combination of individual preference orderings;

(ii) Pareto principle : if every voter prefers A_i to A_j, $i \neq j$, then the collective choice rule should place A_i before A_j in the social ordering;

(iii) independence of irrelevant alternatives : if the original preference orderings of the voters are altered in such a way that each individual's pairwise preferences among a subset of the alternatives remain unchanged, then the social ordering over that subset of the alternatives should be the same as when the original versions of the preference orderings were used;

(iv) nondictatorship : the collective choice rule should not have the property that whenever a specific individual prefers A_i to A_j, $i \neq j$, then A_i is placed before A_j in the social ordering, regardless of the preferences of the other voters.

Employing these minimal requirements, Arrow proved the "rather stunning theorem" (Sen, 1970, p.38) that there is no social welfare function which can simultaneously satisfy all four conditions. Now, although Arrow's impossibility theorem does not apply to social choice functions (Sen, 1970), little comfort is derived from the knowledge that a particular social choice function meets all of Arrow's criteria, because these criteria are so modest and undemanding. If a slightly more demanding set of conditions is erected then an impossibility theorem is also obtained for social choice functions. Thus, consider the following conditions :

(v) anonymity : the social preference should remain unchanged if voters exchange preference orderings with one another;

(vi) neutrality : if the original preference orderings of the voters are altered by interchanging A_i and A_j, $i \neq j$, then it should follow that if A_i is socially preferred to A_j in the original preference orderings then A_j is socially preferred to A_i in the altered preference orderings, and vice versa;

(vii) positive responsiveness : if A_i stands at least as high as A_j, $i \neq j$, in the social preference ordering and if A_i now rises in some voter's preference ordering and does not fall in anyone's preference ordering, then A_i should now be socially preferred to A_j.

Conditions (v), (vi), and (vii) have been shown by Sen (1970) to imply conditions (ii), (iii), and (iv). Therefore, by Arrow's theorem, no social welfare function can simultaneously satisfy conditions (i), (v), (vi), and (vii). However, Sen also achieved a more powerful result, namely, an impossibility theorem for social choice functions. He proved that no social choice function can meet all four of the conditions (i), (v), (vi), and (vii).

Now these four criteria, which together deny the existence of all social choice functions, appear to constitute quite reasonable requirements. However, in the last analysis, a criterion represents a value judgement which may or may not be accepted by everyone. Since individuals may disagree over exactly which criteria ought to be regarded as fundamental, much research has been conducted into the precise conditions satisfied by each collective choice rule, both when used as a social welfare function and when used as a social choice function (e.g. Fishburn, 1973; Richelson, 1975; Young, 1974). With this knowledge, "each individual then can evaluate the desirability of any system according to the relative importance (to the individual) of the conditions satisfied" (Richelson, 1975, p.331, phrase in brackets added). Of course, there is a degree of circularity in this approach, since the need to arrive at a group decision as to which criteria are fundamental would appear to require in advance some form of collective choice rule. However, let us suppose that agreement can be reached on such matters.

A number of criteria have been formulated and analysed in the above manner. The present study offers a further criterion for social choice functions, namely, decisiveness. Other things being equal, the collective choice rule which results in least indecision, that is, which has the lowest likelihood of failure to produce a single clear-cut outcome, is to be preferred to its rivals. However, methods of collective choice generally differ from one another in more than this respect, e.g. the eight criteria examined by Richelson (1975). How might the importance of decisiveness in relation to other criteria be assessed? Clearly, information is required about the probability that each collective choice procedure will result in indecision. If in any given situation this probability is (a) of essentially the same magnitude, or (b) unpredictable over a wide range of possible values, for every collective choice rule then decisiveness will not serve as a useful criterion. On the other hand, if comparative studies reveal that in certain situations, which are relevant to any group, one collective choice procedure is always less susceptible to indecision than another, then decisiveness as a criterion will be difficult to ignore.

Information about the probability of collective indecision is useful in another sphere. Groups are often constitutionally committed to a particular collective choice rule. Knowledge of those situations, e.g. group sizes, in which the possibility of deadlock in the decision-making process is minimised, or maximised, will be to the group's advantage.

To these ends, a comparative analysis is undertaken of the likelihood of indecision under each of the plurality, Condorcet, and Borda procedures. Plurality indecision occurs when there is a winners' tie. Condorcet indecision is synonymous with the paradox of voting when N is odd, and when N is even includes additionally the presence of ties. Borda indecision occurs when there is a winners' tie.

Likelihood approach One way of assessing the gravity of a deficiency in a collective choice rule is to obtain empirical data on the relative frequency with which the deficiency occurs. However, the ubiquity of the plurality method ensures that, in general, only information about first preferences is available. Hence, probabilistic models of collective choice have been developed in order to gauge the likelihood of anomalous outcomes under various collective choice rules. Thus, Niemi and Weisberg (1972, p.391) inform us that "the probability of the paradox of voting was sought only once it was realised that empirical data could not indicate how serious a problem the paradox was for normative democratic theory".

In the present work expressions are developed for the likelihood of the following events :

(a) the paradox of voting ;

(b) agreement between the plurality and Condorcet outcomes;

(c) agreement between the Borda and Condorcet outcomes ;

(d) the Borda reversed - order paradox.

These expressions are used to answer a number of questions of significance for decision - making bodies, e.g. the circumstances under which the likelihood of a given anomaly reaches its maximum, or minimum.

Previous attempts to obtain these likelihood have employed computer search procedures (e.g. Garman and Kamien, 1968 ; Niemi and Weisberg, 1968 ; Paris, 1975), computer simulation routines (e.g. Fishburn, 1974a, 1974b), and explicit, closed - form expressions (e.g. DeMeyer and Plott, 1970). Each approach will be discussed later in connection with the determination of specific likelihoods. However, broadly speaking, it can be said that computer search procedures run into severe time problems, while computer simulation routines cannot guarantee

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1.3 SOCIAL CHOICE AND PSYCHOLOGICAL MEASUREMENT

An interesting by - product of the examination of collective choice rules is the light shed on techniques of data analysis commonly employed by psychologists. Coombs (1954, p.69) comments that "in dealing with their measurement problems, (psychologists) have built formal mechanisms which, while never mentioning 'social utility', actually constitute mechanisms for merging the preferences of the individual members of a group." Since the plurality, Condorcet, and Borda systems are immediately recognisable as analogues of widely practised data analysis procedures, it will be instructive to re- examine the anomalous and paradoxical outcomes of the preceding section, this time in a psychological measurement context.

Paradox of voting. Of Arrow's (1963) four criteria, the Condorcet procedure satisfies the Pareto principle, independence of irrelevant alternatives, and non-dictatorship, but fails to meet the condition of unrestricted domain, because certain combinations of individual preference orderings give rise to the paradox of voting. However, if the condition of unrestricted domain is relaxed by the imposition of a restriction termed "single - peakedness " (Black, 1958 ; Coombs, 1964 ; Sen, 1970) on the set of individual preference orderings, the possibility of the paradox of voting is eliminated. A set of preference orderings is single - peaked if all group members view the alternatives in terms of the same dimension, and that dimension is the only one to influence their judgements about the alternatives. Each individual has a position somewhere along this dimension and arrives at a preference ordering of the alternatives by arranging them according to their absolute distance from his own position, the closest alternative being the most preferred. For example, if individuals I_1 and I_2 and alternatives A_1, A_2, A_3, and A_4 are located on the dimension of political affiliation as follows :

LEFT - WING A_2 ——— I_1 A_4 A_1 I_2 A_3 RIGHT - WING

then the preference orderings of I_1 and I_2 are

I_1 :	A_4, A_1, A_2, A_3	I_2 :	A_3, A_1, A_4, A_2

It is frequently observed in psychology that dependent variables such as "preference, hedonic tone, aesthetic appreciation, stimulus generalisation, degree of interest or attention, exploratory behaviour, developmental stages, and intensity of attitudes" (Coombs and Avrunin, 1977) obey the restriction of single - peakedness. Coombs's (1964) "unfolding theory" relies heavily on the property of single -

peakedness which, he believes, derives from "a common cultural reference frame for communication and evaluation" (Coombs, 1964, p. 397).

Single-peakedness is a special case of the more general condition of "value restriction" (Sen, 1970) which requires that all group members have preference orderings such that, for every set of three (out of m) alternatives, some alternative is (a) not first, or (b) not second, or (c) not third in anyone's ordering of the triple. Single-peakedness corresponds to the third kind of value restriction, i.e. everyone agrees that some alternative is not in third place in the triple. If value restriction holds for every combination of three alternatives then the paradox of voting cannot occur.

The methodological counterpart in psychology of the Condorcet procedure is the technique of paired comparisons (e.g. Bradley, 1968). However, if the results obtained by this technique are used for scaling purposes (e.g. Torgerson, 1958) the similarity between the two procedures disappears. As Coombs (1964, p. 385) remarks, "none of the psychological measurement models are acceptable as social welfare functions, because they all involve interpersonal comparability of utility as a gratuitous assumption". The main difference between the two procedures, as they are commonly employed, is that in the method of paired comparisons individuals do not arrange the alternatives in order of preference, but instead judge the alternatives two at a time, each comparison being made independently of the others. The results for each pair of alternatives are aggregated across individuals, and are usually analysed statistically by means of a test based on the binomial distribution. Because of the additional risk of intransitivity at the level of the individual the method of paired comparisons is more likely to give rise to a cyclical majority than the preference ordering method.

Plurality - Condorcet disagreement. The lengthy association of the plurality procedure with a fundamental democratic procedure, namely, the election of parliamentary representatives, has conferred upon it an aura of fairness and representativeness. In section 1.2 we found this reputation to be undeserved. Consider the following example in which a method parallel to the plurality procedure is used to test a psychological hypothesis.

Example 1.8 After viewing a short film, 100 subjects are presented with five statements, A_1, A_2, A_3, A_4, A_5 and asked to say which one best describes the reasons behind the actions of the principal character. On the basis of attribution theory, the experimenter hypothesises that the first statement, A_1, will be chosen more frequently than any other. The number of subjects actually choosing each statement is found to be

A_1	A_2	A_3	A_4	A_5
34	17	17	16	16

On the basis of the null hypothesis that the statements are equally likely to be chosen, statistical analysis yields a chi - square value of 12.3 with d.f. = 4, which is significant at the .05 level (Bradley, 1968). Since the manner in which the data depart from the null hypothesis accords with his prediction, the experimenter claims that his results provide support for attribution theory. A colleague, however, who is concerned about the merits of the above analysis, obtains from each of the subjects a rank ordering of the statements in terms of how well they describe the reasons behind the actions of the principal character. His results are as follows:

A_1	A_2	A_3	A_4	A_5
A_2	A_3	A_2	A_2	A_2
A_3	A_4	A_4	A_3	A_3
A_4	A_5	A_5	A_5	A_4
A_5	A_1	A_1	A_1	A_1
34	17	17	16	16

Clearly, A_2, A_3, A_4 and A_5 are all regarded as better descriptions than A_1 by 66 subjects to 34 in each case. Statistical analysis using the normal approximation to the binomial distribution yields a standard normal deviate of $z = 3.2$, which in a two-tailed test is significant at the .002 level (Bradley, 1968). Of course, the four comparisons are not independent of one another. However, applying the Bonferroni inequality (Feller, 1968), we can say that A_2, A_3, A_4, and A_5 are all significantly different from A_1 at the 4(.002) or .008 level (two - tailed). Thus, the experimenter's colleague has demonstrated that A_1 is, in fact, regarded as a poorer description than any other statement. Unquestionably, the statement which is regarded as a better description than any other is statement A_2.

By dealing only with first preferences the experimenter is, in effect, testing a special case of his hypothesis. Unless there are a priori grounds for concentrating on first preferences, Example 1.8 makes it clear that such a procedure not only wastes information but can produce thoroughly misleading results.

Borda - Condorcet disagreement and Borda reversed - order paradox. These phenomena are examined together because their underlying causes are related. Consider the following psychological example in which both types of effect may be observed.

Example 1.9 A researcher hypothesises that a person given complimentary feedback by an evaluator on the basis of average task performance is more likely to judge the feedback as sincere the higher the status of the evaluator. An experiment is arranged in which 180 subjects are all given complimentary feedback for "average performance" in each of three comparable tasks. A high status evaluator, A_1, provides the feedback for one of the tasks, a medium status evaluator, A_2, for another, and a low status evaluator, A_3, for the remaining task. Afterwards, A_1, A_2, and A_3 are ranked by the subjects according to the perceived sincerity of their remarks, a rank of 3 being assigned to the most sincere evaluator and a rank of 1 to the least sincere. The results of the experiment are

A_1	A_2
A_3	A_1
A_2	A_3
75	105

Thus, A_1, A_2, and A_3 receive sincerity rank totals of 435, 390, and 255, respectively. Testing for the existence of a monotonic trend by means of the large - sample approximation of the Page test, the experimenter obtains a standard normal deviate of $z = 9.5$, which is significant well beyond the .001 level (Marascuilo and M[c]Sweeney, 1977).

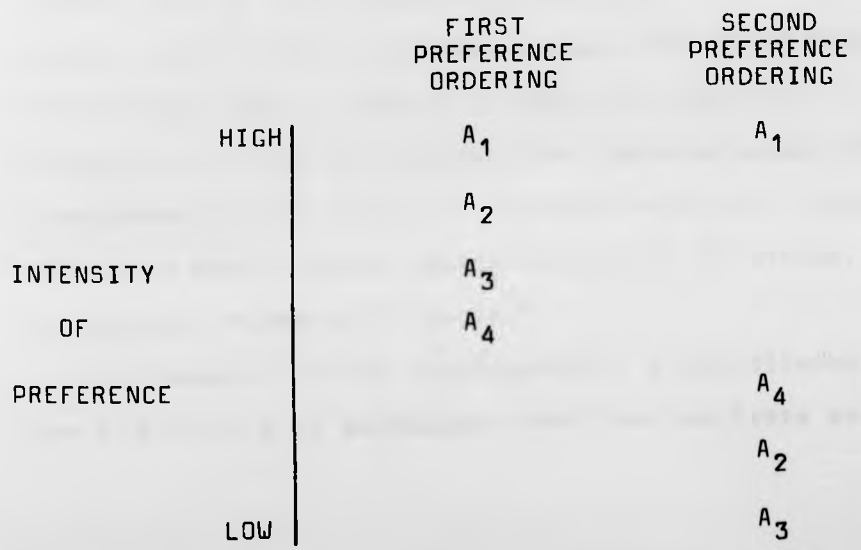
The experimenter concludes that the data support his hypothesis, i.e. complimentary feedback for average performance is regarded as more sincere the higher the status of the evaluator. However, while writing a report on his findings he notices that A_2 is thought more sincere than A_1 by 105 subjects to 75. Statistical analysis using the normal approximation to the binomial distribution yields a standard normal deviate of $z = 2.24$ which in a two-tailed test is significant at the .05 level (Bradley, 1968). The experimenter faces a dilemma. How can the same data produce two apparently contradictory results both of which achieve statistical significance? A colleague advises him to ignore the results of the binomial test on the grounds that a test based on binary values is simply a crude version of a test based on rank order values and consequently does not extract as much information from the data. The experimenter is about to take his colleague's advice when a second crisis occurs. He discovers that for various reasons the evaluator used in the low status condition is unsuitable and must be discarded from the data. His results now look like this

A_1	A_2
A_2	A_1
75	105

Since his last binomial test involved identical figures, he knows that the reduced data indicate that significantly more people regard A_2 as the more sincere evaluator. The experimenter faces a second dilemma. How can removal of the lowest-ranked alternative turn a significant result into one which is significant in the opposite direction?

The experimenter's first dilemma is, of course, structurally identical to the phenomenon of Borda - Condorcet disagreement and his second dilemma is analogous to the Borda reversed - order paradox. The reason that the Borda and Condorcet procedures can disagree is because they ask different questions of the data. At the level of the individual, the Condorcet method asks simply whether A_i is preferred to A_j, whereas the Borda method inquires by how much A_i is preferred to A_j. The Borda method measures the difference in preference intensity between A_i and A_j by means of the number of alternatives in the individual's preference ordering which intervene between A_i and A_j. The larger the number of intervening alternatives the higher A_i's rank relative to A_j's rank (cf. Goodman and Markowitz, 1952). Unfortunately, it cannot be guaranteed that the number of intervening alternatives really does reflect the difference in preference intensity between A_i and A_j. Consider two preference orderings with underlying preference intensities as follows:

FIGURE 1.1



According to the Borda procedure the difference in preference intensity between A_1 and A_4 should be larger in the first preference ordering than in the second. In fact, the reverse is true.

At the level of the group, the Condorcet procedure asks whether the number of people preferring A_i to A_j is larger than the number preferring A_j to A_i. On the other hand, the Borda procedure asks whether the average preference intensity (average rank) recorded by the group members for A_i is higher than that recorded for A_j. Therefore, a second assumption is made by the Borda procedure, namely, interpersonal comparability of preference intensity. Figure 1.1 also illustrates the questionable nature of this assumption.

Describing the philosophy underlying the Condorcet procedure as a theory of relative valuation, and that underlying the Borda procedure as a theory of absolute valuation, Black (1958, p.183) comments :

"A theory of relative valuation affirms nothing about reality which is untrue; people do make relative valuations. This, however, is not the entire matter ..... The theory of relative valuation tells no lie, but it fails to take into account some of the features that characterise valuation in reality, and in this way fails to tell the whole truth.
On the other hand, a theory of absolute valuation, ..., in attempting to take into account the features which are disregarded by the theory of relative valuation, makes crude assertions about reality which we know to be untrue, and incorporates elements of error."

In Example 1.9 the experimenter's first dilemma arises from his failure to appreciate that the two tests are looking at different aspects of the data. From the standpoint of the theory of absolute valuation, although a clear majority perceives A_2 as more sincere than A_1, the intensity with which the minority believes the opposite to be the case enables A_1 to obtain a higher average sincerity rank. The advice of the experimenter's colleague is therefore quite misleading. The problem is not that one test is stronger than the other, but that the tests are examining different hypotheses. The experimenter's second dilemma occurs for the same reason that the Borda reversed - order paradox arises. The Borda method fails to satisfy Arrow's (1963) criterion of independence of irrelevant alternatives. The alternatives intervening between A_i and A_j in a preference ordering are used to provide a measure of the difference in preference intensity between the two. If one of the intervening alternatives is removed the measure is altered. If a non-intervening alternative is removed the measure is unchanged. The reversed - order paradox is generated when this crude system of measurement is employed together with the assumption of interpersonal comparability of preference intensity. Therefore, if he is looking at preference intensities in his data, the experimenter really does face a dilemma when he discards alternative A_3. Of the two contradictory significant findings, either or neither could be correct; such is the imprecision of the measure. On the other hand, if he is looking at the number of people who prefer A_2 to A_1, the experimenter encounters no such difficulties when A_3 is discarded. His conclusion is unaffected.

Finally, if ratings are used instead of ranks the Borda reversed - order paradox cannot occur. However, while it can be argued that at the level of the individual ratings are more accurate than ranks, at the level of the group they still involve the assumption of interpersonal comparability of preference intensity. Also, like ranks, ratings do not supersede paired comparisons as a method of recording the preferences of a group of individuals. On the use of the rating method in voting, Blyth (1972, p.368) comments : "the idea that this 'gives more accuracy' or 'is a refinement', compared to simply asking each voter which of two candidates he prefers ..... is false; as compared with preference statements, scoring is not a refinement, but something quite different."

CHAPTER 2

PROBABILISTIC CHOICE MODEL

2.1 NOTATION AND ASSUMPTIONS

The basic notation set forth below is adhered to throughout the study. To alleviate the need for repeated reference to this section, local notation outlined in each of the next few chapters is supplemented by definitions of relevant symbols from the basic set.

A group of N members wishes to select one of m alternative courses of action $A_1, A_2, \ldots, A_m$. It is assumed that

(i) the alternatives $A_1, A_2, \ldots, A_m$ are mutually exclusive and exhaustive;

(ii) each member arrives at a preference ordering of the alternatives independently of the other group members;

(iii) each member's preference ordering of the alternatives is linear: that is, at the level of the individual, pairwise preferences are transitive, and no member is indifferent between two alternatives;

(iv) each member has probability q_j of choosing preference ordering s_j, where $\Sigma q_j = 1$.

With m alternatives there are m! possible linear preference orderings $s_1, s_2, \ldots, s_{m!}$. The number of members adopting preference ordering s_j is denoted by x_j, where $\Sigma x_j = N$. When m = 3, the 3! = 6 possible preference orderings are as follows:

$s_1 : A_1 > A_2 > A_3$ $\quad$ $s_4 : A_2 > A_3 > A_1$

$s_2 : A_1 > A_3 > A_2$ $\quad$ $s_5 : A_3 > A_1 > A_2$

$s_3 : A_2 > A_1 > A_3$ $\quad$ $s_6 : A_3 > A_2 > A_1$

where $A_f > A_g$ signifies that A_f is preferred to A_g. When m = 4, to prevent confusion, the 4! = 24 possible preference orderings are denoted by $t_1, t_2, \ldots, t_{24}$, and are as follows:

$t_1 : A_1 > A_2 > A_3 > A_4$ $\quad$ $t_{13} : A_3 > A_1 > A_2 > A_4$

$t_2 : A_1 > A_2 > A_4 > A_3$ $\quad$ $t_{14} : A_3 > A_1 > A_4 > A_2$

$t_3 : A_1 > A_3 > A_2 > A_4$ $\quad$ $t_{15} : A_3 > A_2 > A_1 > A_4$

$t_4 : A_1 > A_3 > A_4 > A_2$ $\quad$ $t_{16} : A_3 > A_2 > A_4 > A_1$

$t_5 : A_1 > A_4 > A_2 > A_3$ $\quad$ $t_{17} : A_3 > A_4 > A_1 > A_2$

$t_6 : A_1 > A_4 > A_3 > A_2$ $\quad$ $t_{18} : A_3 > A_4 > A_2 > A_1$

$t_7 : A_2 > A_1 > A_3 > A_4$ $\quad$ $t_{19} : A_4 > A_1 > A_2 > A_3$

$t_8 : A_2 > A_1 > A_4 > A_3$ $\quad$ $t_{20} : A_4 > A_1 > A_3 > A_2$

$t_9 : A_2 > A_3 > A_1 > A_4$	$t_{21} : A_4 > A_2 > A_1 > A_3$
$t_{10} : A_2 > A_3 > A_4 > A_1$	$t_{22} : A_4 > A_2 > A_3 > A_1$
$t_{11} : A_2 > A_4 > A_1 > A_3$	$t_{23} : A_4 > A_3 > A_1 > A_2$
$t_{12} : A_2 > A_4 > A_3 > A_1$	$t_{24} : A_4 > A_3 > A_2 > A_1$

Also, when m = 4 the probability that a member chooses preference ordering t_j is represented by r_j, and the number of members adopting preference ordering t_j is given by y_j. Thus, the terms t_j, r_j, and y_j are counterparts in case m = 4 of the general terms s_j, q_j, and x_j.

Let a_i, b_i, and c_i denote the number of group members with alternative A_i as their first, second, and third preference, respectively. Thus, when m = 3,

$$a_1 = x_1 + x_2 \qquad b_1 = x_3 + x_5 \qquad c_1 = x_4 + x_6$$
$$a_2 = x_3 + x_4 \qquad b_2 = x_1 + x_6 \qquad c_2 = x_2 + x_5$$
$$a_3 = x_5 + x_6 \qquad b_3 = x_2 + x_4 \qquad c_3 = x_1 + x_3 \ .$$

Let p_i denote the probability that a group member chooses A_i as first preference. Thus, when m = 3,

$$p_1 = q_1 + q_2 \qquad p_2 = q_3 + q_4 \qquad p_3 = q_5 + q_6 \ .$$

Let q_{ij} represent the probability that a member prefers A_i to A_j. Thus, when m = 3,

$$q_{12} = q_1 + q_2 + q_5$$
$$q_{13} = q_1 + q_2 + q_3$$
$$q_{23} = q_1 + q_3 + q_4$$

Special symbols.

$\max[e, f, \ldots, g]$	:	maximum of e, f, ..., g
$\min[e, f, \ldots, g]$	:	minimum of e, f, ..., g
$p(e)$	:	probability of e
$p(e\|f)$	:	probability of e given f
$e \cap f$	:	intersection of e and f
e and f	:	intersection of e and f
$\{e\}$	:	integer value of e

2.2 MULTINOMIAL CHOICE MODEL

An essential prerequisite of the analyses to be undertaken in subsequent chapters is an expression for the likelihood of a given distribution of the members over the preference orderings. From the assumptions of the previous section, it follows directly that the probability that $x_1, x_2, \ldots, x_{m!}$ members respectively choose preference orderings $s_1, s_2, \ldots, s_{m!}$ is provided by the multinomial density function (Feller, 1968)

$$p(x_1, x_2, \ldots, x_{m!}) = \frac{N!}{x_1!x_2!\ldots x_{m!}!} q_1^{x_1} q_2^{x_2} \ldots q_{m!}^{x_{m!}} \quad (2.1)$$

If attention is restricted to first preferences, the probability that $a_1, a_2, \ldots, a_m$ members respectively choose $A_1, A_2, \ldots, A_m$ as their first preference is given by

$$p(a_1, a_2, \ldots, a_m) = \frac{N!}{a_1!a_2!\ldots a_m!} p_1^{a_1} p_2^{a_2} \ldots p_m^{a_m} \quad (2.2)$$

A solution for the probability of each of the phenomena investigated in the course of the present study is obtained in the following manner. The problem is formulated in terms of a set of restrictions on the x_j, j = 1, 2, ..., m!, or the a_i, i = 1, 2, ..., m, depending on whether each member's full preference ordering or first preference only is involved. From these restrictions, or inequalities, upper and lower limits for the x_j, or a_i, are derived, thereby defining a region over which the probability function (2.1), or (2.2), may be summed. In this way, a closed - form expression for the likelihood of the phenomenon in question is achieved.

Several researchers (e.g. Davis, 1973; DeMeyer and Plott, 1970; Garman and Kamien, 1968; Gehrlein and Fishburn, 1976; Niemi and Weisberg, 1968, 1972; Paris, 1975) have previously employed the multinomial choice model, in one form or another, to investigate the likelihood of certain collective choice outcomes. A discussion of the strengths and weaknesses of the model may be found in Davis (1973) and in Niemi and Weisberg (1972).

The q_j, j = 1, 2, ..., m!, constitute an internal probability distribution governing the propensity of an individual to adopt possible preference orderings. All members have the same internal probability distribution which is presumed to reflect the "culture" (Garman and Kamien, 1968) or value matrix in which the group is embedded. Thus, individual differences in attitude, motivation, or intensity of preference are not embraced by the model.

Since members act independently of one another, dynamic aspects such as conformity effects, shifting allegiances, etc., are also excluded from consideration. Essentially, we are looking at the internal probability distribution of an individual after cultural values have been absorbed, that is, after social interaction has taken place. Thus, the multinomial choice model "does not deal with the problem of how and why the individual decision distribution comes to have a particular form" (Davis, 1973, p.123).

An implicit assumption of the model is that a member's choice reflects his actual preference. This assumption does not hold when a number of members practise either (a) strategic voting, or (b) logrolling. The former involves a single group decision whereas the latter concerns a series of group decisions made over a period of time. In strategic voting a member's choice contradicts his true preference in order to secure a more favourable outcome (Black, 1958; Niemi and Riker, 1976). In logrolling, or vote - trading, a member exchanges "his vote on one issue for reciprocal support of his own interest by other participants on other issues" (Buchanan and Tullock, 1971, p.121). Thus, neither strategic voting nor logrolling is accommodated by the model.

From the foregoing remarks it is apparent that there are several directions in which the model might be refined and extended. However, as the complexities of the phenomena under scrutiny in the present work unfold in succeeding chapters, it will become clear that the multinomial choice model strikes a balance between the twin demands of providing a close resemblance to reality and facilitating the development of computable solutions.

CHAPTER 3

PLURALITY PROCEDURE

3.1 PROBABILITY A_i IS THE PLURALITY WINNER

The winning alternative under the plurality procedure is the one with the largest share of first preference votes (Rae, 1967). Let $P_m(A_i ; N ; p)$ denote the probability that A_i is the plurality winner when m alternatives are being voted on by N group members in culture p.

Consider alternative A_1. For A_1 to be plurality winner certain relationships must hold among $a_1, a_2, \ldots, a_m$, the number of first preference votes received by $A_1, A_2, \ldots, A_m$, respectively. When $N \leq m$ clearly the lower bound of a_1 occurs when $a_1 = 2$. When $N > m$ and a_1 is at its minimum the maximum number of votes obtainable by each A_j, $j = 2, 3, \ldots, m$, equals $a_1 - 1$. However, from the requirement that $\sum_{i=1}^{m} a_i = N$, it follows that if $a_j = a_1 - 1$, $j = 2, 3, \ldots, m$, then $a_1 + (m - 1)(a_1 - 1) = N$, or $ma_1 = N + m - 1$. Thus, $a_2, a_3, \ldots, a_m$ cannot simultaneously equal $a_1 - 1$ except when $N + m - 1$ is an exact multiple of m. In this event, denote N by N^* and a_1's minimum value by a_1^*. Therefore, $a_1^* = (N^* + m - 1)/m$, an integer value. When $N = N^* + 1$, the minimum value of a_1 must increase by unity to $a_1^* + 1$ since all the a_j, $j = 2, 3, \ldots, m$, are already at their maximum, namely $a_1^* - 1$. Thus, the minimum value of a_1 becomes $(N^* + m - 1)/m + 1$, or $(N + 2(m - 1))/m$, an integer value. This minimum value of a_1 remains the same for all values of N from $N^* + 1$ up to $N^* + m$ when once again $a_2, a_3, \ldots, a_m$ are simultaneously at their maximum, this time a_1^*. Since $(N^* + m) - (N^* + 1)$ divided by m has an integer value of zero, we may write the lower limit of a_1 in general as $a_1 = \{(N + 2(m - 1))/m\}$, where the special brackets signify that the integer value of the expression within is required. To obtain a lower bound for a_2 we observe that $a_1 \geq a_m + 1$, or $a_1 \geq N - \sum_{j=1}^{m-1} a_j + 1$. Therefore, $a_2 \geq N - 2a_1 - \sum_{j=3}^{m-1} a_j + 1$. Now, a_2 reaches its minimum value when both $a_1 = a_m + 1$ and $\sum_{j=3}^{m-1} a_j$ takes its maximum value, i.e. when $a_j = a_1 - 1$, $j = 3, 4, \ldots, m$. In this event, a_2's lower bound equals $N - (m - 1)a_1 + (m - 2)$. However, if $N - a_1$ is not sufficiently large to permit all a_j, $j = 3, 4, \ldots, m$, to equal $a_1 - 1$, then a_2's lower bound equals zero. Accordingly, the lower limit of a_2 is $\max\left[N - (m - 1)a_1 + (m - 2), 0\right]$, where $\max[x, y]$ represents the larger of the values x and y. Similarly, the lower limit of a_3 may be found to equal

$\max\left[N - (m-2)a_1 - a_2 + (m-3) , 0\right]$. In general, the lower limit of a_g, $g = 3, 4, \ldots , m-1$, equals

$$\max\left[N - (m-g+1)a_1 - \sum_{j=2}^{g-1} a_j + (m-g) , 0\right].$$

The upper bound of a_1 is N. The upper bound of a_2 is readily ascertained as $\min\left[a_1 - 1 , N - a_1\right]$, where $\min\left[x , y\right]$ represents the smaller of the values x and y. In general, the upper limit of a_g, $g = 2, 3, \ldots , m-1$, equals

$$\min\left[a_1 - 1 , N - \sum_{j=1}^{g-1} a_j\right].$$

Employing the above results in conjunction with the multinomial social choice model of expression (2.2), we may write

$$P_m(A_1 ; N ; p) = \sum^{m-1} \frac{N!}{a_1!a_2!\ldots a_m!} p_1^{a_1} p_2^{a_2} \ldots p_m^{a_m} , \qquad (3.1)$$

where $\sum^{m-1}$ is an $(m-1)$ - fold summation whose variables of summation have the following limits

$$\begin{aligned}
\left\{(N+2(m-1))/m\right\} &\le a_1 \le N \\
\max\left[N-(m-1)a_1 +(m-2),0\right] &\le a_2 \le \min\left[a_1 -1, N - a_1\right] \\
\max\left[N-(m-2)a_1 -a_2 +(m-3),0\right] &\le a_3 \le \min\left[a_1 -1, N - a_1 - a_2\right] \\
&\ \vdots \\
\max\left[N-(m-g+1)a_1 - \sum_{j=2}^{g-1} a_j +(m-g),0\right] &\le a_g \le \min\left[a_1-1,N - \sum_{j=1}^{g-1} a_j\right] \\
&\ \vdots \\
\max\left[N-2a_1 - \sum_{j=2}^{m-2} a_j +1,0\right] &\le a_{m-1} \le \min\left[a_1-1,N - \sum_{j=1}^{m-2} a_j\right]
\end{aligned} \qquad (3.2)$$

For example, when $m = 4$ the probability that A_1 is the plurality winner is

$$p_4(A_1;\ N;\ p) = \sum^{3} \frac{N!}{a_1!a_2!a_3!a_4!}\ p_1^{a_1}\ p_2^{a_2}\ p_3^{a_3}\ p_4^{a_4}$$ where $\sum^{3}$ is a

3 - fold summation whose variables of summation have the following limits

$$\left\{(N + 6)/4\right\} \leq a_1 \leq N$$

$$\max\left[N - 3a_1 + 2,\ 0\right] \leq a_2 \leq \min\left[a_1 - 1,\ N - a_1\right]$$

$$\max\left[N - 2a_1 - a_2 + 1,\ 0\right] \leq a_3 \leq \min\left[a_1 - 1,\ N - a_1 - a_2\right]$$

Expressions similar to (3.1) and (3.2) may be obtained for the other $p_m(A_i;\ N;\ p)$, $i = 2, 3, \ldots , m$.

3.2 ASYMPTOTIC LIKELIHOOD OF PLURALITY OUTCOMES

In the asymptote, as N becomes very large, the behaviour of $p_m(A_i;\ N;\ p)$ is determined by the magnitude of p_i relative to p_j, all $j \neq i$. If $p_i > p_j$, all $j \neq i$, then $p_m(A_i;\ N;\ p)$ tends to unity as N approaches infinity. On the other hand, if $p_i < p_j$, some $j \neq i$, then $p_m(A_i;\ N;\ p)$ diminishes to zero as N tends to infinity.

To prove this relationship we consider first some of the asymptotic properties of the binomial distribution. Let f represent the number of occurrences in M trials of a binomial random variable with probability of occurrence θ and probability of non-occurrence $1 - \theta$. Define v and w in the unit interval such that $0 < v < w < 1$. The probability that f lies between Mv and Mw is given by

$$p(Mv < f < Mw) = \sum_{f=Mv}^{Mw} \frac{M!}{f!(M - f)!}\ \theta^{f}\ (1 - \theta)^{1-f} \qquad (3.3)$$

As M becomes large the binomial distribution may be approximated by the normal distribution (Feller, 1968 ; Johnson and Kotz ; 1969). In this event, let z_v and z_w denote the standard normal deviates corresponding to Mv and Mw, respectively. Thus,

$$z_v = \frac{Mv - M\theta}{(M\theta(1-\theta))^{1/2}} = \left(\frac{M}{\theta(1-\theta)}\right)^{1/2} (v - \theta) \text{ ,} \tag{3.4}$$

with a similar expression obtaining for z_w. As M tends to infinity, if $(v - \theta) > 0$ then z_v approaches plus infinity, while if $(v - \theta) < 0$ then z_v approaches minus infinity. A similar result holds for z_w. Let $\Phi(z)$ represent the probability under the normal distribution of obtaining a standard normal deviate of z or greater. Thus, when M is large expression (3.3) may be approximated as follows

$$p(Mv < f < Mw) = \Phi(z_v) - \Phi(z_w) \tag{3.5}$$

From the behaviour of z_v and z_w as M tends to infinity, it is apparent that in the asymptote

$$p(Mv < f < Mw) = 1 \quad \text{if } v < \theta < w \text{ ;} \tag{3.6}$$
$$= 0 \quad \text{otherwise.}$$

Expression (3.6) will be of assistance both in the present section and subsequent chapters when asymptotic probabilities are being investigated.

Returning to the plurality procedure, we now consider, in case m = 3, the probability that A_1's share of the plurality votes, a_1, is larger than A_2's share, a_2, when the number of votes received by A_3, a_3, is a fixed proportion g of N, where $0 \leq g < 1$. Thus, $0 \leq a_3 = gN < N$. For A_1 to defeat A_2, the lower limit of a_1 must be h + 1, where $2h = N - a_3$ when $N - a_3$ is even and $2h + 1 = N - a_3$ when $N - a_3$ is odd. Thus, we have

p($a_1 > a_2$ <u>and</u> A_3 has a_3 votes)

$$= \sum_{a_1=h+1}^{N-a_3} \frac{N!}{a_1!a_2!a_3!} p_1^{a_1} p_2^{a_2} p_3^{a_3}$$

$$= \frac{N!}{a_3!(N-a_3)!} p_3^{a_3}(p_1+p_2)^{N-a_3} \sum_{a_1=h+1}^{N-a_3} \frac{(N-a_3)!}{a_1!a_2!}\left(\frac{p_1}{p_1+p_2}\right)^{a_1}\left(\frac{p_2}{p_1+p_2}\right)^{a_2} \tag{3.7}$$

The factor consisting of the terms undergoing aggregation in expression (3.7) constitutes the summation of a binomial density function. From expression (3.6) and from the definition of h, it follows that as N tends to infinity, and likewise therefore $N - a_3$ or $N(1 - g)$, the factor tends to unity when $p_1 > p_2$ and to zero when $p_1 < p_2$. Hence, as N tends to infinity,

p($a_1 > a_2$ **and** A_3 has a_3 votes)

$$= \frac{N!}{a_3!(N-a_3)!} p_3^{a_3} (p_1+p_2)^{N-a_3} \qquad \text{if } p_1 > p_2 ;$$

$$= 0 \qquad \text{if } p_1 < p_2 . \qquad (3.8)$$

Summing (3.8) over all values of a_3, i.e. $0 \leq a_3 < N$, when $p_1 > p_2$ yields

$$p(a_1 > a_2) = \sum_{a_3=0}^{N(.999)} \frac{N!}{a_3!(N-a_3)!} p_3^{a_3} (p_1+p_2)^{N-a_3} \qquad (3.9)$$

where a_3's upper limit equals N multiplied by a value of g arbitrarily close to unity. Therefore, as N tends to infinity,

$$p(a_1 > a_2) = 1 \qquad \text{if } p_1 > p_2 ;$$

$$= 0 \qquad \text{if } p_1 < p_2 . \qquad (3.10)$$

A similar result holds for $p(a_1 > a_3)$. Therefore, as N becomes very large, p($a_1 > a_2$ **and** $a_1 > a_3$) approaches unity if both $p_1 > p_2$ and $p_1 > p_3$ and approaches zero if either $p_1 < p_2$ or $p_1 < p_3$. Similar results occur in the case of p($a_2 > a_1$ **and** $a_2 > a_3$) and p($a_3 > a_1$ **and** $a_3 > a_2$). In other words, as N tends to infinity,

$$p_3(A_i;\ N;\ p) = 1 \qquad \text{if } p_i > p_j,\ \text{all } j \neq i\ ; \qquad (3.11)$$
$$= 0 \qquad \text{if } p_i < p_j,\ \text{some } j \neq i\ .$$

Similar proofs may be invoked when $m > 3$ to demonstrate that A_i is asymptotically certain to become plurality winner provided $p_i > p_j$, all $j \neq i$.

Table 3.1 illustrates in case $m = 3$ the rapidity with which $p_3(A_i;\ N;\ p)$ can converge to unity when p_i is only marginally greater than p_j, all $j \neq i$. The values in the table were calculated by means of expression (3.1).

TABLE 3.1

PROBABILITY THAT A_i IS THE PLURALITY WINNER WHEN THERE ARE N GROUP MEMBERS AND $m = 3$ ALTERNATIVES IN CULTURE p WHERE $p_i = 11/30$, $p_j = 10/30$, AND $p_k = 9/30$.

N	$p_3(A_i;\ N;\ p)$
50	.47182
100	.55516
500	.79267
1000	.89083
5000	.99752

Lastly, an alternative way of expressing the asymptotic probability that A_i is a plurality loser is outlined. The occurrence of the event $a_i < N/m$ implies that an alternative other than A_i is the plurality winner. The probability that $a_i < N/m$ is provided by the binomial distribution

$$p(a_i < N/m) = \sum_{a_i=0}^{N/m} \frac{N!}{a_i!(N-a_i)!}\, p_i^{a_i}\, (1 - p_i)^{N-a_i} \qquad (3.12)$$

From expression (3.6) it is apparent that if $p_i < 1/m$ then $p(a_i < N/m)$ tends to unity as N becomes very large. Therefore, because the event $a_i > N/m$ has zero probability of occurrence when N is large and $p_i < 1/m$, we obtain the asymptotic equality $p(A_i$ is a plurality loser$) = p(a_i < N/m)$, provided $p_i < 1/m$. Thus, as N tends to infinity, we may write

$p(A_i$ is a plurality loser$)$

$$= \sum_{a_i=0}^{N/m} \frac{N!}{a_i!(N-a_i)!} p_i^{a_i} (1 - p_i)^{N-a_i} \text{ if } p_i < 1/m. \qquad (3.13)$$

Expression (3.13) will prove useful in Chapter 7 when the culture in which the likelihood of plurality - Condorcet disagreement reaches its maximum is determined.

3.3 PLURALITY INDECISION

Let $PI_m(N; p)$ represent the probability of plurality indecision when m alternatives are being voted on by N group members in culture p. Since $PI_m(N; p)$ is the probability that a single winning alternative does not emerge, we may write

$$PI_m(N; p) = 1 - \sum_{i=1}^{m} P_m(A_i; N; p) \qquad (3.14)$$

Alternatively, if $PT_m(A_i,A_j; N; p)$ denotes the probability of a plurality winners' tie involving only A_i and A_j, $PT_m(A_i,A_j,A_k; N; p)$ the probability of a plurality winners' tie involving only A_i, A_j and A_k, and so on, then it follows that

$$pI_m(N;\ p) = \sum_{i<j} pT_m(A_i,A_j;\ N;\ p) + \sum_{i<j<k} pT_m(A_i,A_j,A_k;\ N;\ p)$$

$$+ \ldots\ldots + pT_m(A_1,A_2,\ \ldots\ ,A_m;\ N;\ p) \qquad (3.15)$$

With regard to equation (3.14), an expression for $p_m(A_i;\ N;\ p)$ was derived in section 3.1. However, equation (3.15) offers a solution which is much quicker to compute, since the number of ways in which a tie can occur is much smaller than the number of ways in which a win can occur. Accordingly, expressions are developed for $pT_m(A_1,A_2,\ \ldots\ ,A_h;\ N;\ p)$, $h \leq m$. The cases $h = m$, $h = m - 1$, $h = m - 2$, and $h = m - 3$ are considered in turn. The principles outlined in these instances may readily be extended to provide expressions when $h < m - 3$.

If $h = m$, we obtain directly

$$pT_m(A_1,A_2,\ \ldots\ ,A_m;\ N;\ p)$$

$$= \frac{N!}{((N/m)!)^m}\, p_1^{N/m}\, p_2^{N/m}\ \ldots\ p_m^{N/m} \quad \text{if } N \text{ is a multiple of } m;$$

$$= 0 \qquad \text{otherwise.} \qquad (3.16)$$

If $h = m - 1$, the number of votes gained by each of the tying alternatives, $A_1,A_2,\ \ldots\ ,A_{m-1}$, must be greater than the number obtained by the single losing alternative A_m, i.e. $a_1 > a_m$. The lower bound of a_1 depends on the maximum value taken by a_m. When $a_m = a_1 - 1$, it follows from the condition $\sum_{i=1}^{m} a_i = N$ that $(m - 1)a_1 + a_1 - 1 = N$, or $ma_1 - 1 = N$. Therefore, a_m cannot equal $a_1 - 1$ except when $N + 1$ is an exact multiple of m. In this event, denote N by N^* and a_1's minimum by a_1^*. Thus, $a_1^* = (N^* + 1)/m$, an integer value. When $N = N^* + 1$, the minimum of a_1 must increase by unity to $a_1^* + 1$, since a_m is at its maximum $a_1^* - 1$ when $N = N^*$. Thus, the minimum value of a_1 becomes $(N^* + 1)/m + 1$, or $(N + m)/m$, an integer value. However, now

$a_m = N^* + 1 - (m - 1)(a_1^* + 1) = N^* - (m - 1)a_1^* - (m - 2)$

$= a_1^* - 1 - (m - 2) = a_1^* - (m - 1)$. Therefore, a_1's minimum remains the same for all values of N from $N^* + 1$ up to $N^* + m$ when once again $a_m = a_1 - 1$, i.e. $a_m = a_1^*$. Hence, a_1's lower limit may be written in general as $a_1 = \{(N + m)/m\}$, where $\{x\}$ denotes the integer value of x. The upper limit of a_1 is evidently $\{N/(m - 1)\}$. Thus, we have

$$PT_m(A_1, A_2, \ldots, A_{m-1}; N; p) = \sum_{a_1=\left\{\frac{N+m}{m}\right\}}^{\{N/(m-1)\}} \frac{N!}{(a_1!)^{m-1} a_m!} p_1^{a_1} p_2^{a_1} \ldots p_{m-1}^{a_1} p_m^{a_m} \qquad (3.17)$$

where $a_m = N - (m - 1)a_1$. For each combination of m - 1 alternatives an expression similar to (3.17) may be derived for the probability of a winners' tie.

If $h = m - 2$ and $m > 3$, the lower bound of a_1 depends on the maximum value taken by $a_{m-1} + a_m$. When $a_{m-1} = a_m = a_1 - 1$, it follows from the condition $\sum_{i=1}^{m} a_i = N$ that $(m - 2)a_1 + 2(a_1 - 1) = N$, or $ma_1 - 2 = N$. Therefore, a_{m-1} and a_m cannot simultaneously equal $a_1 - 1$ except when $N + 2$ is an exact multiple of m. In this event denote N by N^* and a_1's minimum by a_1^*. Thus, $a_1^* = (N^* + 2)/m$, an integer value. When $N = N^* + 1$, the minimum of a_1 must increase by unity to $a_1^* + 1$, since a_{m-1} and a_m are both at their maximum $a_1^* - 1$ when $N = N^*$. Thus, the minimum value of a_1 becomes $(N^* + 2)/m + 1$, or $(N + m + 1)/m$, an integer value. However, now

$a_{m-1} + a_m = N^* + 1 - (m - 2)(a_1^* + 1) = N^* - (m - 2)a_1^* - (m - 3)$ $= 2(a_1^* - 1) - (m - 3) = 2a_1^* - (m - 1)$. Therefore, a_1's minimum remains the same for all values of N from $N^* + 1$ up to $N^* + m$ when once again $a_{m-1} = a_m = a_1 - 1$ i.e. $a_{m-1} = a_m = a_1^*$. Hence, a_1's lower limit may be written in general as $a_1 = \{(N + m + 1)/m\}$, where $\{x\}$ denotes the integer value of x. The upper bound of a_1 is clearly $\{N/(m - 2)\}$. To determine a_{m-1}'s lower limit we observe that $a_1 \geq a_m + 1$, or $a_1 \geq N - (m - 2)a_1 - a_{m-1} + 1$. Hence, $a_{m-1} \geq N - (m - 1)a_1 + 1$. However, if $N - (m - 2)a_1$ is not sufficiently large to allow a_m to equal $a_1 - 1$ then a_{m-1}'s lower bound equals zero. Thus, $a_{m-1} \geq \max[N - (m - 1)a_1 + 1, 0]$. The upper limit of a_{m-1} is readily seen to equal $\min[a_1 - 1, N - (m - 2)a_1]$. Accordingly,

$$PT_m(A_1, A_2, \ldots, A_{m-2}; N; p) = \sum^{2} \frac{N!}{(a_1!)^{m-2} a_{m-1}! a_m!} p_1^{a_1} \ldots p_{m-2}^{a_1} p_{m-1}^{a_{m-1}} p_m^{a_m} \quad (3.18)$$

where $a_m = N - (m - 2)a_1 - a_{m-1}$, and $\sum^{2}$ is a 2-fold summation whose variables of summation have the following limits :

$$\{(N + m + 1)/m\} \leq a_1 \leq \{N/(m - 2)\}$$
$$\max[N - (m - 1)a_1 + 1, 0] \leq a_{m-1} \leq \min[a_1 - 1, N - (m - 2)a_1]. \quad (3.19)$$

For each combination of m - 2 alternatives expressions similar to (3.18) and (3.19) may be derived for the probability of a winners' tie.

If $h = m - 3$ and $m > 4$, then proceeding as in cases $h = m - 1$ and $h = m - 2$ we establish a_1's lower bound as $\{(N + m + 2)/m\}$ and a_1's upper bound as $\{N/(m - 3)\}$. To determine a_{m-2}'s lower limit we utilise the relationship

$a_1 \geq a_m + 1$, or $a_1 \geq N - (m - 3)a_1 - a_{m-2} - a_{m-1} + 1$. Hence, $a_{m-2} \geq N - (m - 2)a_1 - a_{m-1} + 1$. Now, a_{m-2} reaches its minimum value when a_{m-1} attains its maximum value, i.e. when $a_{m-1} = a_1 - 1$. Therefore, $a_{m-2} \geq N - (m - 1)a_1 + 2$. However, if $N - (m - 3)a_1$ is not sufficiently large to allow all a_j, $j = m - 1, m$, to equal $a_1 - 1$ then the lower bound of a_{m-2} equals zero. Hence, $a_{m-2} \geq \max\left[N - (m - 1)a_1 + 2, 0\right]$. The upper limit of a_{m-2} is readily seen to equal $\min\left[a_1 - 1, N - (m - 3)a_1\right]$. From the relationship established above, namely $a_1 \geq N - (m - 3)a_1 - a_{m-2} - a_{m-1} + 1$, we obtain $a_{m-1} \geq N - (m - 2)a_1 - a_{m-2} + 1$. Thus, reasoning as before, a_{m-1}'s lower limit equals $\max\left[N - (m - 2)a_1 - a_{m-2} + 1, 0\right]$. The upper bound of a_{m-1} evidently equals $\min\left[a_1 - 1, N - (m - 3)a_1 - a_{m-2}\right]$. Hence,

$$PT_m(A_1, A_2, \ldots, A_{m-3}; N; p)$$

$$= \sum^{3} \frac{N!}{(a_1!)^{m-3} a_{m-2}! a_{m-1}! a_m!} p_1^{a_1} \ldots p_{m-3}^{a_1} p_{m-2}^{a_{m-2}} p_{m-1}^{a_{m-1}} p_m^{a_m} \tag{3.20}$$

where $a_m = N - (m - 3)a_1 - a_{m-2} - a_{m-1}$, and $\sum^{3}$ is a 3-fold summation whose variables of summation have the following limits:

$$\begin{aligned}
\left\{(N+m+2)/m\right\} &\leq a_1 \leq \left\{N/(m-3)\right\} \\
\max\left[N-(m-1)a_1+2, 0\right] &\leq a_{m-2} \leq \min\left[a_1-1, N-(m-3)a_1\right] \\
\max\left[N-(m-2)a_1-a_{m-2}+1, 0\right] &\leq a_{m-1} \leq \min\left[a_1-1, N-(m-3)a_1-a_{m-2}\right].
\end{aligned} \tag{3.21}$$

For each combination of m - 3 alternatives expressions similar to (3.20) and (3.21) may be derived for the probability of a winners' tie.

When $h < m - 3$ and $m > 5$ expressions for

$PT_m(A_1, A_2, \ldots, A_h; N; p)$ may be arrived at by a similar chain of reasoning.

As an example of the formulae derived above, when m = 3 the probability of plurality indecision is given by (3.15), (3.16) and (3.17) as

$$
\begin{aligned}
PI_3(N; p) &= PT_3(A_1, A_2; N; p) + PT_3(A_1, A_3; N; p) \\
&\quad + PT_3(A_2, A_3; N; p) \\
&\quad \left(+ PT_3(A_1, A_2, A_3; N; p) \text{ if N is a multiple of 3}\right) \\
&= \sum_{a_1 = \left\{\frac{N+3}{3}\right\}}^{\{N/2\}} \frac{N!}{a_1!a_1!a_3!} \left[p_1^{a_1} p_2^{a_1} p_3^{a_3} + p_1^{a_1} p_2^{a_3} p_3^{a_1} \right. \\
&\qquad \left. + p_1^{a_3} p_2^{a_1} p_3^{a_1} \right] \\
&\quad \left(+ \frac{N}{a_1!a_1!a_1!} p_1^{a_1} p_2^{a_1} p_3^{a_1} \text{ if N is a multiple of 3}\right)
\end{aligned}
\tag{3.22}
$$

where $a_3 = N - 2a_1$. It will be observed that the three pairwise tie probabilities have been brought together under a single summation sign with a_1 as the variable of summation. Since the number of votes received by each member of a pairwise winners' tie has the same range of values irrespective of the identities of the alternatives making up the pair, the values taken by a_1 in the above expression correspond to the number of votes received by any member of a pairwise winners' tie. Likewise, the values taken by a_3 correspond to the number of votes received by any losing alternative when the other two form a winners' tie.

3.4 EFFECT OF GROUP SIZE, NUMBER OF ALTERNATIVES, AND CULTURE ON THE LIKELIHOOD OF PLURALITY INDECISION

Using the formulae derived in section 3.3, the probability of plurality indecision, $PI_m(N; p)$, was calculated for a number of values of N, m, and p. Consider Table 3.2 which provides the probability of plurality indecision as N and m vary and p remains constant at equiprobability.

As might be expected the likelihood of plurality indecision decreases toward zero as N becomes very large, irrespective of the number of alternatives. Moreover, the rate of approach to zero is approximately the same for $m = 3$, 4, and 5. The likelihood of plurality indecision declines sharply to begin with and changes to a gentler pace of descent around $N = 100$. Also, as the number of alternatives increases so, broadly speaking, does the probability of indecision.

In general terms, therefore, a decision making body employing the plurality procedure in an equiprobable culture becomes less effective as group size diminishes and as the number of alternatives being put to the vote increases. Indeed, groups with less than 20 members have a rather high potential for indecision, at least once out of every eight voting occasions. In particular, committees, which often consist of fewer than ten members, are in this culture certain to equivocate on at least a sixth of the occasions in which a vote is taken on three or more options.

For a given group size, as the number of alternatives increases while maintaining cultural equiprobability, the likelihood of indecision would appear to increase toward unity,

TABLE 3.2

PROBABILITY OF PLURALITY INDECISION FOR N GROUP MEMBERS AND m ALTERNATIVES UNDER AN EQUIPROBABLE CULTURE p IN WHICH $p_i = 1/m$, $i = 1, 2, \ldots, m$.

N	m = 3	m = 4	m = 5
3	.22222	.37500	.48000
4	.22222	.23438	.28800
5	.37037	.35156	.32640
6	.20576	.38086	.41600
7	.19204	.25635	.37632
8	.28807	.24994	.29210
9	.18137	.28519	.28278
10	.17284	.28182	.30755
11	.24254	.22823	.31158
12	.16430	.22790	.28482
13	.15822	.24164	.25732
14	.21311	.23876	.25859
15	.15140	.20778	.26536
16	.14680	.20682	.26113
17	.19212	.21435	.24655
18	.14122	.21256	.23391
19	.13758	.19134	.23436
20	.17621	.19023	.23580
50	.10879	.13308	.15973
100	.06764	.09532	.11753
500	.03332	.04454	.05555
1000	.02262	.03181	.03981

though clearly this takes very much longer with larger group sizes. The point may be illustrated by considering first a group of three members. When N = 3, plurality indecision occurs only if each member chooses a different one of the m alternatives. The probability of this occurrence is

$$PI_m(3;\ p) = \binom{m}{3}\frac{3!}{m^3} = \frac{(m-1)(m-2)}{m^2} \quad \text{if } p_i=1/m, \text{ all } i. \qquad (3.23)$$

Expression (3.23) tends to unity as m becomes large. In general, when $m \geq N$, the situation in which each member chooses a different one of the m alternatives, is usually only one of the ways in which plurality indecision can occur. Thus, we have

$$PI_m(N;\ p) \geq \binom{m}{N}\frac{N!}{m^N} = \frac{m(m-1)\ldots.(m-N+1)}{m^N} \quad \text{if } p_i=1/m, \text{ all } i, \text{ and if } m \geq N. \qquad (3.24)$$

Expression (3.24) also tends to unity as m becomes very large. Thus, when the number of alternatives is very much larger than the number of voters the likelihood of plurality indecision in an equiprobable culture can approach unity.

Perhaps most intriguing of all in this culture is the nonmonotonic behaviour of $PI_m(N;\ p)$ both in relation to group size and in relation to number of alternatives, especially when N and m are both small. Thus, when m = 3, a four-person committee is as likely to fail to reach a decision as a six-person committee, i.e. once out of every five occasions. However, a five-person committee has virtually double that chance of failure.

So far we have examined the effects on $PI_m(N;\ p)$ of group size and number of alternatives when culture p remains constant at equiprobability. If culture p departs from equiprobability then the asymptotic results of section 3.2 indicate that $PI_m(N; p)$ tends to zero as N becomes very large. Table 3.3 illustrates the behaviour of $PI_m(N; p)$ in various non-equiprobable cultures for values of N ranging from small to moderately large, with the number of alternatives constant at $m = 3$.

For given m, it is clear that a culture may be found in which $PI_m(N; p)$ is at its highest for a particular value of N. An interesting question is whether a culture exists in which plurality indecision is at a maximum for all values of N. The equiprobable culture comes close to fulfilling this role especially as m becomes larger. However, when $m = 3$ the equiprobable culture results in the maximum likelihood of plurality indecision only for all odd values of N and for $N = 8$ and $N = 14$. Otherwise, culture (i) in Table 3.3, where $p = (.5, .5, 0)$, produces the maximum probability of indecision for most even values of N. When $m = 4$ and $m = 5$, once again the equiprobable culture is the one which gives rise to the greatest likelihood of indecision, this time for all values of N except $N = 4$, 5, 8 and 11 when $m = 4$, and $N = 4$, 5 and 9 when $m = 5$.

Knowledge of the upper limit of the likelihood of plurality indecision for given N and m can be useful to a group. It enables the group to decide on the size of its membership which will minimise the effects of plurality indecision should the culture most productive of plurality indecision prevail. For example, on the basis of such a "minimax" strategy, the optimum size of a committee required to consist of less than ten members, who would be deciding among three, four, or five alternatives at a time, would be nine members.

TABLE 3.3

PROBABILITY OF PLURALITY INDECISION FOR N GROUP MEMBERS AND $m = 3$ ALTERNATIVES IN EACH OF THREE CULTURES : (i) $p = (.5, .5, 0)$, (ii) $p = (.5, .25, .25)$, AND (iii) $p = (.8, .1, .1)$

N	Culture (i)	Culture (ii)	Culture (iii)
3	.00000	.18750	.04800
4	.37500	.21094	.07740
5	.00000	.29297	.04080
6	.31250	.17090	.02626
7	.00000	.15381	.01445
8	.27344	.20615	.01183
9	.00000	.13298	.00603
10	.24609	.12377	.00425
11	.00000	.15377	.00282
12	.22559	.10756	.00172
13	.00000	.10059	.00111
14	.20947	.11981	.00076
15	.00000	.08817	.00046
16	.19638	.08285	.00030
17	.00000	.09566	.00020
18	.18547	.07305	.00013
19	.00000	.06885	.00008
20	.17620	.07764	.00006
49	.00000	.01344	.00000
50	.11228	.01371	.00000
99	.00000	.00115	.00000
100	.07959	.00110	.00000

CHAPTER 4

CONDORCET PROCEDURE

4.1 PROBABILITY A_i IS THE CONDORCET WINNER

The collective choice under the Condorcet procedure is arrived at by examining the alternatives two at a time and counting the number of individual preference orderings in which one of the alternatives in a pair is ranked higher than the other. From each pairwise comparison a simple majority winner emerges. The Condorcet winner is the alternative, if one exists, which defeats all the others in pairwise simple majority contests (Black, 1958). Let $C_m(A_i; N; q)$ denote the probability that A_i is the Condorcet winner when m alternatives are being voted on by N group members in culture q.

Previous studies have employed a variety of methods to calculate $C_m(A_i; N; q)$. All, however, assume the same multinomial social choice model. Klahr (1966) and Pomerantz and Weil (1970) estimated $C_m(A_i; N; q)$ in the equiprobable case by means of computer simulation. Garman and Kamien (1968) and Niemi and Weisberg (1968) devised computer search procedures to procure those distributions of the voters on the preference orderings which give rise to a Condorcet winner, and were able to obtain exact values for $C_m(A_i; N; q)$. As the computational labour required by these approaches is immense for anything other than relatively modest values of N and m, methods for approximating $C_m(A_i; N; q)$ using the multivariate normal approximation to the multinomial were developed by Garman and Kamien (1968), May (1971), and Niemi and Weisberg (1968). The multivariate normal approximation was also used by Gleser (1969) to set upper and lower bounds for $C_m(A_i; N; q)$. The first analytic, closed-form expression for $C_m(A_i; N; q)$ was developed by DeMeyer and Plott (1970). A considerably simpler solution was obtained by Gillett (1976,1977) and independently by Gehrlein and Fishburn (1976). Although the latter two solutions share essentially the same train of reasoning, Gillett's is simpler and quicker to compute. Nevertheless, in comparison with earlier work both solutions constitute a major advance, especially with larger m. Thus, the calculation of $C_5(A_i; 5; q)$, where q represents the equiprobable culture, was estimated by DeMeyer and Plott (1970) to require 300 hours computing time, but was achieved by Gehrlein and Fishburn (1976) in a computing time of less than one second.

Three alternatives. When m = 3 there are 3! = 6 possible linear preference orderings $s_1, s_2, \ldots, s_6$ as follows:

$s_1 : A_1 > A_2 > A_3 \qquad s_4 : A_2 > A_3 > A_1$

$s_2 : A_1 > A_3 > A_2 \qquad s_5 : A_3 > A_1 > A_2$

$s_3 : A_2 > A_1 > A_3 \qquad s_6 : A_3 > A_2 > A_1$

where $A_i > A_j$ signifies that A_i is preferred to A_j. The probability that a voter chooses preference ordering s_h is given by q_h, where $\Sigma\, q_h = 1$. The set of preference ordering probabilities $q = (q_1, q_2, ..., q_6)$ is presumed to mirror the culture of which the group forms a part. The number of voters choosing preference ordering s_h is represented by x_h, where $\Sigma x_h = N$.

Focussing our attention on alternative A_1, we proceed by grouping together those preference orderings in which A_1's position relative to the other alternatives is the same. Let F_1 denote the number of voters placing A_1 in last position in their preference orderings; let F_2 and F_3 denote the number of voters with preference orderings s_3 and s_5 respectively; lastly, let F_4 denote the number of voters with A_1 as their first preference. Thus,

$$F_1 = x_4 + x_6; \quad F_2 = x_3; \quad F_3 = x_5; \quad F_4 = x_1 + x_2.$$

The probability that a voter places A_1 last in his preference ordering equals $q_4 + q_6$; preference orderings s_3 and s_5 are chosen by a voter with probabilities q_3 and q_5 respectively; and the probability that a voter chooses A_1 as first preference equals $q_1 + q_2$.

For A_1 to be the Condorcet winner it is required that A_1 defeat A_2 and A_3 in simple majority contests. That is the following inequalities must hold

$$\begin{aligned} F_1 + F_3 &< F_2 + F_4 \\ F_1 + F_2 &< F_3 + F_4 \quad , \end{aligned} \tag{4.1}$$

or equivalently

$$\begin{aligned} F_1 + F_3 &\leq k \\ F_1 + F_2 &\leq k \quad , \end{aligned} \tag{4.2}$$

where $2k + 1 = N$ when N is odd and $2k + 2 = N$ when N is even.

Clearly each of F_1, F_2, and F_3 has a lower limit of zero which may be reached independently of the values of the other two. The upper limit of F_1 is k. For a given value of F_1 the upper limit of F_2 equals $k - F_1$. For given values of F_1 and F_2 the upper limit of F_3 depends solely on the value of F_1 and therefore also equals $k - F_1$.

Employing these results in conjunction with the multinomial social choice model, we obtain a closed - form expression for the probability that A_1 is the Condorcet winner. That is

$$C_3(A_1; N; q) = \sum^{3} \frac{N!}{F_1!F_2!F_3!F_4!} (q_4 + q_6)^{F_1} q_3^{F_2} q_5^{F_3} (q_1 + q_2)^{F_4} \quad (4.3)$$

where $\sum^{3}$ is a 3 - fold summation whose variables of summation have the following limits

$$\begin{aligned} 0 &\leq F_1 \leq k \\ 0 &\leq F_2 \leq k - F_1 \\ 0 &\leq F_3 \leq k - F_1 \ . \end{aligned} \quad (4.4)$$

The solution embodied by expressions (4.3) and (4.4) is computationally much less arduous than that of any previous approach, e.g. DeMeyer and Plott (1970). However, it admits of further simplification.

The area of summation defined by (4.4) may be split into two mutually exclusive and exhaustive regions as follows:

$$\begin{aligned} 0 &\leq F_1 \leq k \\ 0 &\leq F_2 \leq k - F_1 \\ 0 &\leq F_3 \leq k - F_1 - F_2 \end{aligned} \quad (4.5)$$

and

$$
\begin{aligned}
0 &\leq F_1 \leq k-1 \\
1 &\leq F_2 \leq k-F_1 \\
k-F_1-F_2+1 &\leq F_3 \leq k-F_1
\end{aligned}
\tag{4.6}
$$

Region (4.5) represents all permutations of the values of F_1, F_2, and F_3 such that $F_1 + F_2 + F_3 = F$, where $0 \leq F \leq k$. The multinomial density function in expression (4.3) may be rewritten as

$$\frac{N!}{F!(N-F)!} \cdot \frac{F!}{F_1!F_2!F_3!} (q_4 + q_6)^{F_1} q_3^{F_2} q_5^{F_3} (q_1 + q_2)^{N-F} \tag{4.7}$$

Summing expression (4.7) over all permutations of the values of F_1, F_2, and F_3 such that $F_1 + F_2 + F_3 = F$, we obtain by the multinomial theorem the following expression

$$\frac{N!}{F!(N-F)!} (q_3 + q_4 + q_5 + q_6)^F (q_1 + q_2)^{N-F} \tag{4.8}$$

It remains but to sum F from zero to k, whereupon we have succeeded in simplifying expression (4.3), summed over region (4.5), to the cumulative binomial

$$\sum_{F=0}^{k} \frac{N!}{F!(N-F)!} (1 - q_1 - q_2)^F (q_1 + q_2)^{N-F} \tag{4.9}$$

No such simplification seems possible when expression (4.3) is summed over region (4.6). Hence, we have finally

$$
\begin{aligned}
C_3(A_1; N; q) = &\sum_{F=0}^{k} \frac{N!}{F!(N-F)!} (1 - q_1 - q_2)^F (q_1 + q_2)^{N-F} \\
&+ \sum^{3} \frac{N!}{F_1!F_2!F_3!F_4!} (q_4 + q_6)^{F_1} q_3^{F_2} q_5^{F_3} (q_1 + q_2)^{F_4}
\end{aligned}
\tag{4.10}
$$

where $\sum^{3}$ is a 3 - fold summation whose variables of summation have limits given by (4.6).

Similar expressions may be obtained for $C_3(A_2; N; q)$

and $C_3(A_3; N; q)$.

A rough assessment of the computational advantage of expression (4.10) over expression (4.3) may be obtained by comparing the number of terms being summed in both solutions. In the area of summation defined by (4.4) there are $(k - F_1 + 1)^2$ summations for each value of F_1. Let $i = k - F_1 + 1$. Thus, region (4.4) involves

$$\sum_{i=1}^{k+1} i^2 = (k + 1)(k + 2)(2k + 3)/6$$ summations. In region (4.5), effectively, k entities are being distributed among four categories. The number of summations equals the number of possible distributions, namely $\binom{k+3}{k} = (k + 3)(k + 2)(k + 1)/6$. The number of summations in region (4.6) is found by subtracting the number in region (4.5) from the number in region (4.4), which gives $k(k + 1)(k + 2)/6$ summations. Thus, the number of terms being summed in expression (4.10) is $k(k + 1)(k + 2)/6 + (k + 1) = (k + 1)(k^2 + 2k + 6)/6$, and the number in expression (4.3) is $(k + 1)(k + 2)(2k + 3)/6$. The computational advantage of (4.10) over (4.3) may be expressed as the ratio of these two quantities, or $(k^2 + 2k + 6) : (2k^2 + 7k + 6)$. From this ratio it is apparent that, for virtually all values of k, expression (4.10) involves less than half the computational labour demanded by expression (4.3).

<u>Four alternatives.</u> When $m = 4$ there are $4! = 24$ possible linear preference orderings $t_1, t_2, \ldots, t_{24}$ as follows:

t_1	$: A_1 > A_2 > A_3 > A_4$	t_{13}	$: A_3 > A_1 > A_2 > A_4$
t_2	$: A_1 > A_2 > A_4 > A_3$	t_{14}	$: A_3 > A_1 > A_4 > A_2$
t_3	$: A_1 > A_3 > A_2 > A_4$	t_{15}	$: A_3 > A_2 > A_1 > A_4$
t_4	$: A_1 > A_3 > A_4 > A_2$	t_{16}	$: A_3 > A_2 > A_4 > A_1$
t_5	$: A_1 > A_4 > A_2 > A_3$	t_{17}	$: A_3 > A_4 > A_1 > A_2$
t_6	$: A_1 > A_4 > A_3 > A_2$	t_{18}	$: A_3 > A_4 > A_2 > A_1$
t_7	$: A_2 > A_1 > A_3 > A_4$	t_{19}	$: A_4 > A_1 > A_2 > A_3$
t_8	$: A_2 > A_1 > A_4 > A_3$	t_{20}	$: A_4 > A_1 > A_3 > A_2$
t_9	$: A_2 > A_3 > A_1 > A_4$	t_{21}	$: A_4 > A_2 > A_1 > A_3$
t_{10}	$: A_2 > A_3 > A_4 > A_1$	t_{22}	$: A_4 > A_2 > A_3 > A_1$
t_{11}	$: A_2 > A_4 > A_1 > A_3$	t_{23}	$: A_4 > A_3 > A_1 > A_2$
t_{12}	$: A_2 > A_4 > A_3 > A_1$	t_{24}	$: A_4 > A_3 > A_2 > A_1$

where $A_i > A_j$ signifies that A_i is preferred to A_j. The probability that a voter chooses preference ordering t_h is given by r_h, where $\Sigma r_h = 1$. The set of preference ordering probabilities $r = (r_1, r_2, \ldots, r_{24})$ is presumed to reflect the culture in which the group is embedded. The number of voters choosing preference ordering t_h is represented by y_h, where $\Sigma y_h = N$.

Consider alternative A_1. As in the case of three alternatives, our basic strategy is to group together those preference orderings in which A_1's position relative to the other alternatives is the same. Thus, define

$$D_1 = y_{10} + y_{12} + y_{16} + y_{18} + y_{22} + y_{24} \quad ;$$
$$D_2 = y_{17} + y_{23} \quad ;$$
$$D_3 = y_{19} + y_{20} \quad ;$$
$$D_4 = y_{11} + y_{21} \quad ;$$
$$D_5 = y_{13} + y_{14} \quad ;$$
$$D_6 = y_7 + y_8 \quad ;$$
$$D_7 = y_9 + y_{15} \quad ;$$
$$D_8 = y_1 + y_2 + y_3 + y_4 + y_5 + y_6 \quad ;$$

and, likewise, define

$$u_1 = r_{10} + r_{12} + r_{16} + r_{18} + r_{22} + r_{24} \quad ;$$
$$u_2 = r_{17} + r_{23} \quad ;$$
$$u_3 = r_{19} + r_{20} \quad ;$$
$$u_4 = r_{11} + r_{21} \quad ;$$
$$u_5 = r_{13} + r_{14} \quad ;$$
$$u_6 = r_7 + r_8 \quad ;$$
$$u_7 = r_9 + r_{15} \quad ;$$
$$u_8 = r_1 + r_2 + r_2 + r_4 + r_5 + r_6 \quad .$$

It follows that $\Sigma D_j = N$ and $\Sigma u_j = 1$.

For A_1 to be Condorcet winner it is required that A_1 defeat each of A_2, A_3, and A_4 in simple majority contests. That is, the following inequalities must hold

$$\begin{aligned} D_1 + D_4 + D_6 + D_7 &< D_2 + D_3 + D_5 + D_8 \\ D_1 + D_2 + D_5 + D_7 &< D_3 + D_4 + D_6 + D_8 \qquad (4.11) \\ D_1 + D_2 + D_3 + D_4 &< D_5 + D_6 + D_7 + D_8 \end{aligned}$$

or equivalently

$$D_1 + D_4 + D_6 + D_7 \leq k$$
$$D_1 + D_2 + D_5 + D_7 \leq k \qquad (4.12)$$
$$D_1 + D_2 + D_3 + D_4 \leq k$$

where $2k + 1 = N$ when N is odd and $2k+2 = N$ when N is even.

Evidently, each of D_1, D_2, D_3, D_4, D_5, D_6, and D_7 has a lower limit of zero which may be reached independently of the values of the other six. The upper limit of D_1 is k. For a given value of D_1 the upper bound of D_2 equals $k - D_1$. For given D_1 and D_2 the upper limit of D_3 equals $k - D_1 - D_2$. For given D_1, D_2, and D_3 the upper bound of D_4 equals $k - D_1 - D_2 - D_3$. For given D_1, D_2, D_3, and D_4, because $N - D_1 - D_2 - D_3 - D_4 > k$, the upper limit of D_5 is influenced only by D_1 and D_2 and so equals $k - D_1 - D_2$. For given D_1, D_2, D_3, D_4, and D_5 it is required that $D_6 \leq N - D_1 - D_2 - D_3 - D_4 - D_5$ and also that $D_6 \leq k - D_1 - D_4$. Since either of these limits may be smaller than the other depending on whether $N - k - D_2 - D_3 - D_5$ is a positive or a negative quantity, the lower bound of D_6 is written as $\min\left[N - D_1 - D_2 - D_3 - D_4 - D_5,\ k - D_1 - D_4\right]$. Lastly, for given D_1, D_2, D_3, D_4, D_5, and D_6 it is required that $D_7 \leq N - D_1 - D_2 - D_3 - D_4 - D_5 - D_6$, that $D_7 \leq k - D_1 - D_2 - D_5$, and that $D_7 \leq k - D_1 - D_4 - D_6$. Once again any of these limits may be smallest depending on the values of D_1, D_2, ..., D_6. Hence, the lower bound of D_7 equals $\min\left[N - D_1-D_2-D_3-D_4-D_5-D_6,\ k - D_1-D_2-D_5,\ k - D_1-D_4-D_6\right]$.

Application of these findings to the multinomial social choice model yields an expression for the probability that A_1 is the Condorcet winner. Thus,

$$C_4(A_1;\ N;\ r) = \sum^{7} \frac{N!}{D_1!D_2!\ldots.D_8!} u_1^{D_1}\ u_2^{D_2} \ldots.\ u_8^{D_8} \tag{4.13}$$

where $\sum^{7}$ is a 7-fold summation whose variables of summation have the following limits

$$\begin{aligned}
0 &\leq D_1 \leq k \\
0 &\leq D_2 \leq k - D_1 \\
0 &\leq D_3 \leq k - D_1 - D_2 \\
0 &\leq D_4 \leq k - D_1 - D_2 - D_3 \\
0 &\leq D_5 \leq k - D_1 - D_2 \\
0 &\leq D_6 \leq \min\left[N-D_1-D_2-D_3-D_4-D_5,\ k-D_1-D_4\right] \\
0 &\leq D_7 \leq \min\left[N-D_1-D_2-D_3-D_4-D_5-D_6,\ k-D_1-D_2-D_5,\ k-D_1-D_4-D_6\right]
\end{aligned} \tag{4.14}$$

Proceeding as in case m = 3 the area of summation delimited by (4.14) may be divided into two mutually exclusive and exhaustive regions:

$$\begin{aligned}
0 &\leq D_1 \leq k \\
0 &\leq D_2 \leq k - D_1 \\
0 &\leq D_3 \leq k - D_1 - D_2 \\
0 &\leq D_4 \leq k - D_1 - D_2 - D_3 \\
0 &\leq D_5 \leq k - D_1 - D_2 - D_3 - D_4 \\
0 &\leq D_6 \leq k - D_1 - D_2 - D_3 - D_4 - D_5 \\
0 &\leq D_7 \leq k - D_1 - D_2 - D_3 - D_4 - D_5 - D_6
\end{aligned} \tag{4.15}$$

and

$$
\begin{aligned}
0 &\leq D_1 \leq k - 1 \\
0 &\leq D_2 \leq k - D_1 \\
0 &\leq D_3 \leq k - D_1 - D_2 \\
0 &\leq D_4 \leq k - D_1 - D_2 - D_3 \\
\max\left[0, 1-D_2-D_3\right] &\leq D_5 \leq k - D_1 - D_2 \\
\max\left[0, 1-D_3-D_4\right] &\leq D_6 \leq \min\left[N-D_1-D_2-D_3-D_4-D_5, k-D_1-D_4\right] \\
\max\left[0, k+1-D_1-D_2-D_3 -D_4-D_5-D_6\right] &\leq D_7 \leq \min\left[N-D_1-D_2-D_3-D_4-D_5-D_6, k-D_1-D_2-D_5, k-D_1-D_4-D_6\right].
\end{aligned}
\tag{4.16}
$$

By a chain of reasoning parallel to that followed in the case of three alternatives, expression (4.13) summed over region (4.15) reduces to a cumulative binomial. Hence, if

$D = \sum_{i=1}^{7} D_i$, expression (4.13) summed over region (4.14) becomes finally

$$
C_4(A_1; N; r) = \sum_{D=0}^{k} \frac{N!}{D!(N-D)!} (1 - u_8)^D u_8^{N-D}
$$

$$
+ \sum^{7} \frac{N!}{D_1!D_2!\ldots.D_8!} u_1^{D_1} u_2^{D_2} \ldots. u_8^{D_8} \tag{4.17}
$$

where $\sum^{7}$ is a 7-fold summation whose variables of summation have limits given by (4.16).

Similar expressions may be obtained for $C_4(A_2; N; r)$, $C_4(A_3; N; r)$, and $C_4(A_4; N; r)$.

Because of the complex nature of the area of summation defined by (4.14) it was not possible to obtain a ratio, as in the case of three alternatives, expressing the computational advantage of (4.17) over (4.13). However, from a comparison of computing times, it turns out that the reduction in computational labour effected by (4.17) over (4.13), although considerable in absolute terms, is not so large in relative terms as was the reduction achieved in case $m = 3$.

Five or more alternatives. The principles outlined in cases $m = 3$ and $m = 4$ may readily be extended to permit the development of formulae for $C_m(A_i; N; q)$ when $m \geq 5$.

Gehrlein and Fishburn (1976) independently arrived at formulae very similar to those of (4.3) and (4.13) although in the latter case their arrangement of the variables of summation was quite different. At this point their investigation halted. However, as demonstrated, expressions (4.3) and (4.13) may be simplified to expressions (4.10) and (4.17) respectively, thereby achieving an additional saving in computational labour. In the next section an approach is developed which produces simplifications of a still greater order.

4.2 A RECURSION RELATION FOR THE PROBABILITY THAT A_i IS THE CONDORCET WINNER

In the special case of the equiprobable culture, May (1971) and Gehrlein and Fishburn (1976) obtained recursion relations permitting the calculation of $C_m(A_i; N; q)$, when N is odd and m is even, as a linear combination of $C_j(A_i; N; q)$ for odd $j < m$. In this section a recursion relation is developed which provides an expression for $C_m(A_i; N; q)$ in terms of $C_m(A_i; N - 1; q)$, that is, recursion takes place over N instead of over m. Slightly different solutions are required depending on whether N is even or odd. The relationship outlined holds for all cultures q, is readily generalised to all m, and permits a substantial saving in computational time over the solutions presented in the previous section.

<u>N odd</u> Let $2k + 1 = N$. If A_i defeats A_j in a simple majority sense when the number of voters $N - 1$ is even, then the advent of one additional voter cannot change this outcome, because the smallest margin of victory in a simple majority contest is always two votes when there is an even number of voters.

Consider the case of $m = 3$ alternatives. If A_1 is the Condorcet winner when the number of voters $N - 1$ is even, in which case, by the inequalities (4.2), $F_1 + F_2 \leq k - 1$ and $F_1 + F_3 \leq k - 1$, then the arrival of the N^{th} voter will not alter this state of affairs. On the other hand, if A_1 is not the Condorcet winner when the number of voters $N - 1$ is even, there are three mutually exclusive and exhaustive situations in which the arrival of the N^{th} voter could bring about the emergence of A_1 as Condorcet winner: (i) whenever a majority prefers A_1 to A_2 but A_1 and A_3 tie with k votes apiece, i.e. $F_1 + F_2 \leq k - 1$ and $F_1 + F_3 = k$; (ii) whenever a majority prefers A_1 to A_3 but A_1 and A_2 tie, i.e. $F_1 + F_2 = k$ and $F_1 + F_3 \leq k - 1$; and (iii) whenever A_1 ties with A_2 and A_1 ties with A_3, i.e. $F_1 + F_2 = k$ and $F_1 + F_3 = k$. Should the N^{th} voter prefer A_1 to A_3 in the first situation, A_1 to A_2 in the second situation, or both A_1 to A_2 and A_1 to A_3 in the third situation, then A_1 becomes the Condorcet winner. The probability that a voter prefers A_1 to A_3 equals $q_1 + q_2 + q_3$, that he prefers A_1 to A_2

equals $q_1 + q_2 + q_5$, and that he prefers both A_1 to A_2 and A_1 to A_3 equals $q_1 + q_2$. Such considerations lead directly to the following recursion relation for the probability that A_1 is the Condorcet winner when N is odd:

$$\begin{aligned} C_3(A_1; \text{N odd}; q) = {} & C_3(A_1; N-1; q) \\ & + (q_1+q_2+q_3)\, p(F_1+F_2 \leq k-1 \ \underline{\text{and}}\ F_1+F_3 = k; N-1) \\ & + (q_1+q_2+q_5)\, p(F_1+F_2 = k \ \underline{\text{and}}\ F_1+F_3 \leq k-1; N-1) \\ & + (q_1+q_2)\, p(F_1+F_2 = k \ \underline{\text{and}}\ F_1 + F_3 = k; N-1) \end{aligned} \tag{4.18}$$

where p(x) denotes the probability of x.

The terms of the form p(x) may be evaluated by means of the expression on the right hand side of equation (4.3) with certain adjustments to the limits of summation in (4.4). To this end, it is useful to employ G_1, G_2, and G_3 as variables of summation in place of F_1, F_2, and F_3.

Both $p(F_1 + F_2 \leq k - 1$ and $F_1 + F_3 = k; N-1)$ and $p(F_1 + F_2 = k$ and $F_1 + F_3 \leq k - 1; N-1)$ have the following limits of summation:

$$\begin{aligned} 0 \leq {} & G_1 \leq k-1 \\ 0 \leq {} & G_2 \leq k-1-G_1 \\ & G_3 = k - G_1 \end{aligned} \tag{4.19}$$

where $F_1 = G_1$, $F_2 = G_2$, and $F_3 = G_3$ in the former case, and $F_1 = G_1$, $F_2 = G_3$, and $F_3 = G_2$ in the latter.

The limits of summation for $p(F_1 + F_2 = k$ and $F_1 + F_3 = k; N-1)$ are

$$0 \leq G_1 \leq k$$

$$G_2 = k - G_1 \qquad (4.20)$$

$$G_3 = k - G_1$$

where $F_i = G_i$, $i = 1, 2, 3$.

Substituting in (4.18) and rearranging terms, we obtain finally

$$C_3(A_1 ; \text{N odd}; q) = C_3(A_1; N-1; q)$$

$$+ \sum_{G_1=0}^{k-1} \sum_{G_2=0}^{k-1-G_1} \frac{(N-1)!}{G_1!G_2!(k-G_1)!(k-G_2)!} (q_4+q_6)^{G_1}$$

$$\cdot (q_1+q_2)^{k-G_2} \Big[(q_1+q_2+q_3)q_3^{G_2} q_5^{k-G_1}$$

$$+ (q_1 + q_2 + q_5)q_3^{k-G_1} q_5^{G_2} \Big]$$

$$+ \sum_{G_1=0}^{k} \frac{(N-1)!}{G_1!(k-G_1)!(k-G_1)!G_1!} (q_4+q_6)^{G_1} q_3^{k-G_1}$$

$$\cdot q_5^{k-G_1} (q_1 + q_2)^{G_1+1} \qquad (4.21)$$

Similar relationships may be obtained for $C_3(A_2; \text{N odd}; q)$ and $C_3(A_3; \text{N odd}; q)$.

A rough approximation to the computational advantage of (4.21) over (4.3) may be found by comparing the number of terms being summed in each expression. It was established in section 4.1 that there are $(k + 1)(k + 2)(2k + 3)/6$ terms in expression (4.3). If it is assumed that $C_3(A_1; N - 1; q)$ is known, the number of terms being summed in (4.21) equals $k(k + 1)/2 + (k + 1) = (k + 1)(k + 2)/2$. The computational advantage of (4.21) over (4.3) may be represented by the ratio of their respective numbers of terms, that is

$(k + 1)(k + 2)/2 : (k + 1)(k + 2)(2k + 3)/6$, which simplifies to $3 : (2k + 3)$ or around $3 : N$ when N is large. For example, if $N = 300$ expression (4.21) involves only one hundredth of the labour demanded by expression (4.3).

In the case of four alternatives, define

$$D_{21} = D_1 + D_4 + D_6 + D_7 ;$$
$$D_{31} = D_1 + D_2 + D_5 + D_7 ;$$
$$D_{41} = D_1 + D_2 + D_3 + D_4 .$$

Evidently, D_{21}, D_{31}, and D_{41} represent the number of voters preferring A_2 to A_1, A_3 to A_1, and A_4 to A_1, respectively. A recursion relation for $C_4(A_1; N \text{ odd}; r)$ is arrived at by a process of reasoning similar to that employed in the case of three alternatives. Thus,

$$\begin{aligned} C_4(A_1; N \text{ odd}; r) = C_4(A_1; N{-}1; r) \\ &+ (u_8+u_5+u_3+u_2)p(D_{21}=k, D_{31}\leq k-1, D_{41}\leq k-1; N-1) \\ &+ (u_8+u_6+u_4+u_3)p(D_{21}\leq k-1, D_{31}=k, D_{41}\leq k-1; N-1) \\ &+ (u_8+u_7+u_6+u_5)p(D_{21}\leq k-1, D_{31}\leq k-1, D_{41}=k; N-1) \\ &+ (u_8+u_3)p(D_{21}=k, D_{31}=k, D_{41}\leq k-1; N-1) \\ &+ (u_8+u_5)p(D_{21}=k, D_{31}\leq k-1, D_{41}=k; N-1) \\ &+ (u_8+u_6)p(D_{21}\leq k-1, D_{31}=k, D_{41}=k; N-1) \\ &+ (u_8)p(D_{21}=k, D_{31}=k, D_{41}=k; N-1) \end{aligned} \qquad (4.22)$$

The terms of the form $p(x)$ may be evaluated by means of the expression on the right hand side of equation (4.13) with certain adjustments to the limits of summation (4.14). To this end, it is useful to employ $E_1, E_2, \ldots, E_7$ as variables of summation in place of $D_1, D_2, \ldots, D_7$.

An expression for $p(D_{21}=k, D_{31} \leq k-1, D_{41} \leq k-1; N-1)$ requires that $D_7 = k - D_1 - D_4 - D_6$. Since $D_1 + D_2 + D_5 + D_7 \leq k - 1$ it follows that $D_6 \geq 1 + D_2 + D_5 - D_4$. If $D_4 > 1 + D_2 + D_5$ we have $D_6 \geq 0$. Thus, the lower limit of D_6 equals $\max\left[1 + D_2 + D_5 - D_4,\ 0\right]$. As $D_2 + D_3 + D_5 + D_8 = k$, for given values of D_1, D_2, D_3, and D_4 it follows that $D_5 \leq k - D_2 - D_3$, thereby altering the upper bound of D_5 to $\min\left[k - 1 - D_1 - D_2,\ k - D_2 - D_3\right]$. Because $N - 1 - D_1 - D_2 - D_3 - D_4 - D_5$ is always greater than $k - D_1 - D_4$ the upper limit of D_6 becomes simply $k - D_1 - D_4$. Accordingly, $p(D_{21}=k, D_{31} \leq k-1, D_{41} \leq k-1; N-1)$ has the following limits of summation:

$$
\begin{aligned}
0 &\leq E_1 \leq k - 1 \\
0 &\leq E_2 \leq k - 1 - E_1 \\
0 &\leq E_3 \leq k - 1 - E_1 - E_2 \\
0 &\leq E_4 \leq k - 1 - E_1 - E_2 - E_3 \qquad (4.23)\\
0 &\leq E_5 \leq \min\left[k - 1 - E_1 - E_2,\ k - E_2 - E_3\right] \\
\max\left[1 + E_2 + E_5 - E_4,\ 0\right] &\leq E_6 \leq k - E_1 - E_4 \\
E_7 &= k - E_1 - E_4 - E_6
\end{aligned}
$$

where $D_i = E_i$, $i = 1, 2, \ldots, 7$. The limits of summation for $p(D_{21} \leq k-1, D_{31}=k, D_{41} \leq k-1; N-1)$ and for $p(D_{21} \leq k-1, D_{31} \leq k-1, D_{41}=k; N-1)$ are also provided by (4.23) with in the former case

$$
\begin{aligned}
&D_1 = E_1 \quad D_3 = E_5 \quad D_5 = E_6 \quad D_7 = E_4 \\
&D_2 = E_7 \quad D_4 = E_2 \quad D_6 = E_3
\end{aligned}
$$

and in the latter

$$
\begin{aligned}
&D_1 = E_1 \quad D_3 = E_6 \quad D_5 = E_3 \quad D_7 = E_2 \\
&D_2 = E_4 \quad D_4 = E_7 \quad D_6 = E_5 \quad .
\end{aligned}
$$

When calculating $p(D_{21}=k, D_{31}=k, D_{41} \leq k-1;\ N-1)$ the limits of summation may be further simplified. From the two equalities, $D_{21} = k$ and $D_{31} = k$, it is apparent that $D_6 = D_2 + D_5 - D_4$. Because D_6 cannot take a negative value but may equal zero for any given values of D_1, D_2, D_3, and D_4, it follows that $D_2 + D_5 \geq D_4$ or $D_5 \geq D_4 - D_2$, giving a lower bound for D_5 of $\max\left[D_4 - D_2,\ 0\right]$. For given D_1, D_2, D_3, and D_4, it is evident from the two equalities $D_{21} = k$ and $D_{31} = k$, that the upper bound of D_5 equals $\min\left[k - D_1 - D_2,\ k - D_2 - D_3\right]$. Thus, we obtain the following limits of summation for $p(D_{21}=k, D_{31}=k, D_{41} \leq k-1;\ N-1)$:

$$
\begin{aligned}
0 &\leq E_1 \leq k - 1 \\
0 &\leq E_2 \leq k - 1 - E_1 \\
0 &\leq E_3 \leq k - 1 - E_1 - E_2 \\
0 &\leq E_4 \leq k - 1 - E_1 - E_2 - E_3 \\
\max\left[E_4 - E_2,\ 0\right] &\leq E_5 \leq \min\left[k - E_1 - E_2,\ k - E_2 - E_3\right] \\
E_6 &= E_2 + E_5 - E_4 \\
E_7 &= k - E_1 - E_4 - E_6
\end{aligned}
\qquad (4.24)
$$

where $D_i = E_i$, $i = 1, 2, \ldots, 7$. The limits of summation for $p(D_{21}=k, D_{31} \leq k-1, D_{41}=k;\ N-1)$ and for $p(D_{21} \leq k-1, D_{31}=k, D_{41}=k;\ N-1)$ are also provided by (4.24) with in the former case

$$
\begin{aligned}
D_1 &= E_1 & D_3 &= E_6 & D_5 &= E_3 & D_7 &= E_2 \\
D_2 &= E_4 & D_4 &= E_7 & D_6 &= E_5 & &
\end{aligned}
$$

and in the latter

$$
\begin{aligned}
D_1 &= E_1 & D_3 &= E_5 & D_5 &= E_6 & D_7 &= E_4 \\
D_2 &= E_7 & D_4 &= E_2 & D_6 &= E_3 & & .
\end{aligned}
$$

Lastly, the limits of summation of

$p(D_{21}=k, D_{31}=k, D_{41}=k;\ N-1)$ are

$$
\begin{aligned}
0 \le\ & E_1 \le k \\
0 \le\ & E_2 \le k - E_1 \\
0 \le\ & E_3 \le k - E_1 - E_2 \\
& E_4 = k - E_1 - E_2 - E_3 \\
\max\left[E_4 - E_2,\ 0\right] \le\ & E_5 \le \min\left[k - E_1 - E_2,\ k - E_2 - E_3\right] \\
& E_6 = E_2 + E_5 - E_4 \\
& E_7 = k - E_1 - E_4 - E_6
\end{aligned}
\qquad (4.25)
$$

where $D_i = E_i$, $i = 1, 2, \ldots, 7$.

Similar recursion relations may be obtained for $C_4(A_2;\ N\text{ odd};\ r)$, $C_4(A_3;\ N\text{ odd};\ r)$ and $C_4(A_4;\ N\text{ odd};\ r)$.

The complex nature of the regions of summation delimited by (4.14), (4.23), (4.24), and (4.25) makes it difficult to arrive at a simple ratio expressing the computational advantage of the recursion relation (4.22) over expression (4.13), as was done in the case of three alternatives. However, inspection does suggest that a reduction in the number of terms required to be summed has been achieved. This view is supported by a comparison of the times taken to compute $C_4(A_1;\ N\text{ odd};\ r)$, firstly using (4.13), and secondly employing (4.22) when $C_4(A_1;\ N - 1;\ r)$ is known. For example, when $N = 33$ the latter method requires one quarter of the time taken by the former.

As the principle involved in the generation of the recursion relation is easily extended to cover the cases of five or more alternatives, further expressions need not be developed.

<u>N even.</u> Let $2k + 2 = N$. If A_i defeats A_j in a simple majority sense when the number of voters $N - 1$ is odd, then the advent of one additional voter can change this outcome only if

the margin of victory is a single vote, in which case the new arrival may vote for A_j thus causing a tie.

Consider the case of three alternatives. If A_1 is the Condorcet winner when the number of voters N - 1 is odd, in which case, by the inequalities (4.2), $F_1 + F_2 \leq k$ and $F_1 + F_3 \leq k$, then there are three mutually exclusive and exhaustive situations in which the arrival of the N^{th} voter could prevent A_1 from continuing to be the Condorcet winner: (i) whenever A_1's majority over A_2 is greater than a single vote, but A_1's majority over A_3 equals one vote only, i.e. $F_1 + F_2 \leq k - 1$ and $F_1 + F_3 = k$; (ii) whenever A_1's majority over A_3 is larger than one vote, but A_1's majority over A_2 consists of a single vote, i.e. $F_1 + F_2 = k$ and $F_1 + F_3 \leq k - 1$; and (iii) whenever A_1's majority over A_2 equals one vote and A_1's majority over A_3 also equals one vote, i.e. $F_1 + F_2 = k$ and $F_1 + F_3 = k$. Should the N^{th} voter prefer A_3 to A_1 in the first situation (with probability $q_4 + q_5 + q_6$), A_2 to A_1 in the second situation (with probability $q_3 + q_4 + q_6$), or either A_3 to A_1 or A_2 to A_1 or both in the third situation (with probability $q_3 + q_4 + q_5 + q_6$), then A_1 would no longer be the Condorcet winner. Hence, we have the following recursion relation for the probability that A_1 is the Condorcet winner when N is even:

$$
\begin{aligned}
C_3(A_1;\ N\ \text{even};\ q) = C_3(A_1;\ N\text{-}1;\ q)
& - (q_4+q_5+q_6)p(F_1+F_2\leq k\text{-}1\ \underline{\text{and}}\ F_1+F_3=k;\ N\text{-}1) \\
& - (q_3+q_4+q_6)p(F_1+F_2=k\ \underline{\text{and}}\ F_1+F_3\leq k\text{-}1;\ N\text{-}1) \\
& - (q_3+q_4+q_5+q_6)p(F_1+F_2=k\ \underline{\text{and}}\ F_1+F_3=k;\ N\text{-}1)
\end{aligned} \tag{4.26}
$$

The terms of the form p(x) may be evaluated by means of the expression on the right hand side of equation (4.3) with limits of summation identical to those provided by (4.19) and (4.20) in case N odd. Therefore

$$C_3(A_1;\ N\ \text{even};\ q) = C_3(A_1;\ N-1;\ q)$$

$$- \sum_{G_1=0}^{k-1} \sum_{G_2=0}^{k-1-G_1} \frac{(N-1)!}{G_1!G_2!(k-G_1)!(k+1-G_1)!} (q_4+q_6)^{G_1}$$

$$\cdot (q_1+q_2)^{k+1-G_1} \Big[(q_4+q_5+q_6) q_3^{G_2} q_5^{k-G_1}$$

$$+ (q_3+q_4+q_6) q_3^{k-G_1} q_5^{G_2} \Big]$$

$$- (q_3+q_4+q_5+q_6) \sum_{G_1=0}^{k} \frac{(N-1)!}{G_1!(k-G_1)!(k-G_1)!(G_1+1)!}$$

$$\cdot (q_4+q_6)^{G_1} q_3^{k-G_1} q_5^{k-G_1} (q_1+q_2)^{G_1+1} \tag{4.27}$$

The bounds of summation in expression (4.27) when $2k + 2 = N$ are identical to those in expression (4.21) when $2k + 1 = N - 1$, i.e. if $N - 1$ is substituted for N as the odd number of voters in (4.21). The summands in (4.21) and (4.27) are also highly similar. Thus, $C_3(A_1;\ N-1\ \text{odd};\ q)$ and $C_3(A_1;\ N\ \text{even};\ q)$ may be computed simultaneously in almost the same time required to calculate one of them on its own.

Similar expressions may be obtained for $C_3(A_2;\ N\ \text{even};\ q)$ and $C_3(A_3;\ N\ \text{even};\ q)$.

As in the case of odd N, the computational advantage of (4.27) over (4.3) is given approximately by the ratio 3 : N. That is, if $C_3(A_1;\ N-1;\ q)$ is known, then expression (4.27) may be computed in a fraction 3/N of the time taken by expression (4.3).

In the case of four alternatives, by a similar chain of reasoning as in the case of three alternatives, a recursion relation for the probability that A_1 is the Condorcet winner may be developed. Thus,

$$\begin{aligned} C_4(A_1;\ N\ \text{even};\ r) = {} & C_4(A_1;\ N-1;\ r) \\ & - (1-u_8-u_5-u_3-u_2)p(D_{21}=k, D_{31}\leq k-1, D_{41}\leq k-1;\ N-1) \\ & - (1-u_8-u_6-u_4-u_3)p(D_{21}\leq k-1, D_{31}=k, D_{41}\leq k-1;\ N-1) \\ & - (1-u_8-u_7-u_6-u_5)p(D_{21}\leq k-1, D_{31}\leq k-1, D_{41}=k;\ N-1) \\ & - (1-u_8-u_3)p(D_{21}=k, D_{31}=k, D_{41}\leq k-1;\ N-1) \\ & - (1-u_8-u_5)p(D_{21}=k, D_{31}\leq k-1, D_{41}=k;\ N-1) \\ & - (1-u_8-u_6)p(D_{21}\leq k-1, D_{31}=k, D_{41}=k;\ N-1) \\ & - (1-u_8)p(D_{21}=k, D_{31}=k, D_{41}=k;\ N-1) \end{aligned} \qquad (4.28)$$

The terms of the form p(x) may be evaluated by means of the expression on the right hand side of equation (4.13) with limits of summation identical to those provided by (4.23), (4.24), and (4.25) in case N odd, except for the upper bound of E_5 where $k - E_2 - E_3$ is replaced by $k + 1 - E_2 - E_3$.

The computational advantage of (4.28) over (4.13) is the same as that of (4.22) over (4.13) in case N odd.

Generalisation of the recursion relation for $m > 4$ is straightforward, though ascertaining the appropriate bounds of summation for the component parts of the relationship can be a somewhat laborious task. When $m > 4$, as N increases the computational advantage of the recursion relation also increases though with considerably slower rates of growth as m becomes larger.

4.3 EFFECT OF GROUP SIZE, NUMBER OF ALTERNATIVES, AND CULTURE ON THE LIKELIHOOD OF CONDORCET INDECISION

Let $CI_m(N; q)$ denote the probability of Condorcet indecision when N group members vote on m alternatives in culture q. As Condorcet indecision occurs whenever a Condorcet winner fails to emerge, we have

$$CI_m(N; q) = 1 - \sum_{i=1}^{m} C_m(A_i; N; q) \qquad (4.29)$$

When N is odd, Condorcet indecision can arise only in the event of a cyclical majority. When N is even, either the existence of a cyclical majority or the presence of a tie in one or more of the pairwise contests may induce Condorcet indecision. All previous studies, being concerned mainly with the cyclical majority problem, have dealt exclusively with the case of odd-sized groups, in which the probability of the paradox of voting is more easily derived. As indecision, whatever its source, is of interest in the present work the cases of both odd - and even-sized groups are treated below.

The asymptotic behaviour of $CI_m(N; q)$, both as N becomes large and as m becomes large, has received much scrutiny in the special case of the equiprobable culture, i.e. $q_j = 1/m!$, $j = 1, 2, \ldots, m!$ (Garman and Kamien, 1968; May, 1971; Niemi and Weisberg, 1968). For given N, $CI_m(N; q)$ tends to unity as m becomes large, though "the limit is attained too slowly to be of real concern to behavioural scientists" (May, 1971). When m = 3, as N becomes large $CI_3(N; q)$ asymptotes at .08774, or Guilbaud's number as it has come to be designated (Garman and Kamien, 1968). Other limiting values are $CI_4(\infty; q) = .17548$ and $CI_5(\infty; q) = .25131$.

The equiprobable culture has been described by May (1971) as "quite pathological" in the sense that the slightest deviation from precise equiprobability totally alters the above asymptotic results. To this extent the equiprobable culture is an unrealistic one, and of less concern than non-equiprobable cultures. The asymptotic behaviour of $CI_m(N; q)$ in non-equiprobable cultures has been studied by Gleser (1969) and May (1971). The next two paragraphs achieve much the same results as these authors but by simpler means. Moreover, the principles outlined below will prove useful in a later chapter when the asymptotic behaviour of the likelihood of plurality distortion is considered.

Let q_{ij} denote the probability that a voter prefers A_i to A_j. Thus, in the case of three alternatives $q_{12} = q_1 + q_2 + q_5$, and in the case of four alternatives $q_{12} = u_2 + u_3 + u_5 + u_8$. Let V be the number of voters with preference orderings in which A_i stands higher than A_j. For example, if i = 1 and j = 2, then when m = 3 we have $V = x_1 + x_2 + x_5$, and when m = 4 we have $V = D_2 + D_3 + D_5 + D_8$. Hence, the likelihood that A_i is socially preferred to A_j is given by

$$p(A_i \text{ more votes than } A_j) = \sum_{V=k+1}^{N} \binom{N}{V} q_{ij}^{V} (1 - q_{ij})^{N-V} \qquad (4.30)$$

where 2k + 1 = N if N is odd and 2k = N if N is even. By the result contained in expression (3.6), it follows that if $q_{ij} > 1/2$ then $p(A_i$ more votes than $A_j)$ tends to unity as N becomes large . Table 4.1 incorporates exact probabilities computed by means of (4.30) for a culture in which q_{ij} is only marginally greater than 1/2. The limiting value of unity is reached fairly quickly in spite of the fact that a voter is only very slightly more likely to prefer A_i.

TABLE 4.1

PROBABILITY THAT A_i RECEIVES MORE VOTES THAN A_j WHEN THERE ARE N GROUP MEMBERS IN CULTURE q WHERE $q_{ij} = 11/21$.

N	p(A_i more votes than A_j)
50	.57823
100	.64718
500	.84642
1000	.92995
5000	.99960

Now, when each of a number of events has a probability of occurrence equal to unity (zero), then the probability of their joint occurrence must also equal unity (zero).

As $C_m(A_i;\ N;\ q) = p(\bigcap_{j \neq i} [A_i \text{ more votes than } A_j])$, where $\bigcap_{j \neq i} [w_j]$ denotes the intersection of the w_j for all $j \neq i$, it follows that as N becomes large

$$C_m(A_i;\ N;\ q) = 1 \quad \text{if } q_{ij} > 1/2, \text{ all } j \neq i; \qquad (4.31)$$
$$= 0 \quad \text{if } q_{ij} < 1/2, \text{ some } j \neq i.$$

Also, since $CI_m(N;\ q) = 1 - \sum_{i=1}^{m} C_m(A_i;\ N;\ q)$ it is clear that as N becomes large

$$CI_m(N;\ q) = 0 \quad \text{if there is an } A_i \text{ with } q_{ij} > 1/2, \text{ all } j \neq i; \qquad (4.32)$$
$$= 1 \quad \text{if there is no } A_i \text{ with } q_{ij} > 1/2, \text{ all } j \neq i.$$

For example, in the case of three alternatives, the culture $q = (0, .2, .1, .1, .4, .2)$ yields $q_{12} = 0 + .2 + .4 = .6$, $q_{31} = .1 + .4 + .2 = .7$, and $q_{32} = .2 + .4 + .2 = .8$. Therefore, as N becomes large, A_3 is virtually certain to defeat both A_1

and A_2 and so to become the Condorcet winner. By contrast culture $q = (.3, .1, 0, .3, .2, .1)$ yields $q_{12} = .3 + .1 + .2 = .6$, $q_{23} = .3 + 0 + .3 = .6$, and $q_{31} = .3 + .2 + .1 = .6$. As N becomes large, A_1 is almost certain to receive more votes than A_2, A_2 more votes than A_3, and A_3 more votes than A_1, thus making the paradox of voting virtually inevitable.

Turning our attention to more moderate sizes of N and m, the formulae derived in sections 4.1 and 4.2 were used to calculate the probability of Condorcet indecision, $CI_m(N; q)$, for a number of values of N, m, and q. In Table 4.2 the likelihood of Condorcet indecision is provided for $3 \leq m \leq 5$ and $3 \leq N \leq 20$ while q remains constant at equiprobability.

Particularly striking is the difference between groups with odd and even numbers of members. It will be recalled that while odd-sized groups are susceptible to one form of indecision, namely the occurrence of the paradox of voting, even-sized groups can encounter two types of indecision, i.e. the occurrence of the paradox of voting or the presence of ties in some of the pairwise contests. From Table 4.2 it is apparent that in an equiprobable culture the likelihood of indecision caused by ties is rather large. Indeed, for a given number of alternatives, every odd-sized group is less likely than every even-sized group to result in Condorcet indecision. Thus, a committee employing the Condorcet procedure can dramatically increase or decrease its prospects of effectiveness by expanding or reducing its membership by a single person. For example, in an equiprobable culture when $m = 3$, loss of a member through illness from a committee of six would mean that, instead of one in every two ballots producing an ambiguous outcome, only one in every 14 now would do so.

For given N, the likelihood of Condorcet indecision

TABLE 4.2

PROBABILITY OF CONDORCET INDECISION FOR N GROUP MEMBERS AND m ALTERNATIVES UNDER AN EQUIPROBABLE CULTURE q IN WHICH $q_i = 1/m!$, $i = 1, 2, \ldots, m!$.

N	m = 3	m = 4	m = 5
3	.05556	.11111	.16000
4	.55556	.65799	.72325
5	.06944	.13889	.19953
6	.49126	.59992	.67184
7	.07502	.15003	.21533
8	.44810	.55919	.63465
9	.07798	.15595	.22372
10	.41660	.52866	.60624
11	.07981	.15963	.22892
12	.39230	.50466	.58361
13	.08107	.16214	.23247
14	.37280	.48515	.56503
15	.08198	.16396	.23504
16	.35670	.46886	.54940
17	.08267	.16534	.23700
18	.34311	.45499	.53603
19	.08322	.16643	.23854
20	.33143	.44299	.52444

increases monotonically with m. By m = 5 every even-sized group of 20 or fewer members is more likely than not to give rise to an equivocal result. Thus, in contrast to May's (1971) findings with odd-sized groups, as m increases the probability of Condorcet indecision in the case of even-sized groups approaches the asymptote of unity fairly rapidly. The reason for this difference is quite simply the higher initial value of the even likelihood; the rate of growth with m of the odd likelihood is in fact the faster of the two.

For given m in the equiprobable culture, as N becomes large, odd- and even-sized groups approach the limiting value in different manners. Consider the case of three alternatives. Table 4.3 represents an extension of Table 4.2 in case m = 3. Odd-sized groups start off below the asymptotic value of .08774 but fairly close to it, and they rise quickly to meet it.

TABLE 4.3

PROBABILITY OF CONDORCET INDECISION FOR N GROUP MEMBERS AND m = 3 ALTERNATIVES IN AN EQUIPROBABLE CULTURE p.

N	$CI_3(N; p)$
99	.08688
100	.20297
499	.08757
500	.14039
999	.08766
1000	.12516

Even-sized groups, on the other hand, begin high above the asymptotic value and, although falling consistently, are still further away from the asymptote at group size 1000 than is the initial odd-sized group of three members. Thus, when $m = 3$ it would appear that, in the equiprobable culture, every odd-sized group is more decisive than every even-sized group, for all N. A similar result is to be expected when $m > 3$.

So far we have studied the likelihood of Condorcet indecision for moderate sizes of N and m in the case of the equiprobable culture only. Table 4.4 contains values of $CI_m(N; q)$ when $m = 3$ for each of three different cultures. Culture (i) in which $q = (1/4, 0, 1/4, 1/4, 0, 1/4)$ and culture (ii) in which $q = (1/3, 0, 1/3, 0, 1/3, 0)$ both have the property of single-peakedness discussed in chapter 1. Hence, they do not give rise to the paradox of voting. Any indecision in these cultures is due entirely to the presence of ties. For this reason the likelihood of Condorcet indecision in these cultures is zero for odd-sized groups. In both cultures the probability of indecision tends to zero as N increases, the limiting value being attained by $N = 100$ in culture (i) and shortly thereafter in culture (ii). Culture (iii) where $q = (1/3, 0, 0, 1/3, 1/3, 0)$ is the culture in which the likelihood of Condorcet indecision reaches its maximum for all values of N. As N increases the probability of indecision in this culture tends to unity, the asymptote being reached, once again, around $N = 100$.

In every culture we have examined there has been a considerable odd/even discrepancy in the likelihood of Condorcet indecision when N is small. This result is not unexpected. From the considerations advanced in the development of a recursion relation for $C_m(A_i; N; q)$, it is apparent that when N is odd $C_m(A_i; N; q) > C_m(A_i; N-1; q)$ and when N is even the reverse is true. Therefore, when N is odd $CI_m(N; q) < CI_m(N-1; q)$. Nevertheless, while the greater

TABLE 4.4

PROBABILITY OF CONDORCET INDECISION FOR N GROUP MEMBERS AND m = 3 ALTERNATIVES IN EACH OF THREE CULTURES:

(i) q = (1/4, 0, 1/4, 1/4, 0, 1/4),

(ii) q = (1/3, 0, 1/3, 0, 1/3, 0), AND

(iii) q = (1/3, 0, 0, 1/3, 1/3, 0).

N	Culture (i)	Culture (ii)	Culture (iii)
3	.00000	.00000	.22222
4	.39844	.51852	.66666
5	.00000	.00000	.37037
6	.25879	.41152	.69959
7	.00000	.00000	.48011
8	.17197	.33074	.73617
9	.00000	.00000	.56546
10	.11656	.26886	.77031
11	.00000	.00000	.63374
12	.08024	.22081	.80066
13	.00000	.00000	.68938
14	.05591	.18297	.82715
15	.00000	.00000	.73531
16	.03932	.15278	.85011
17	.00000	.00000	.77357
18	.02785	.12838	.86996
19	.00000	.00000	.80570
20	.01984	.10847	.88709
49	.00000	.00000	.97638
50	.00017	.01182	.98525
99	.00000	.00000	.99907
100	.00000	.00044	.99940

decisiveness of odd-sized groups is known a priori, the size of the odd/even discrepancy in small groups is somewhat surprising. As N increases the discrepancy progressively diminishes. Clearly, small decision making groups employing the Condorcet procedure can achieve greater efficacy simply by adopting a size restriction of an odd number of members.

CHAPTER 5

BORDA PROCEDURE

5.1 PROBABILITY A_i IS THE BORDA WINNER

In the Borda social choice procedure, or method of marks as it is sometimes known, a group member's ordering of the m alternatives from most preferred to least preferred is used as a basis for allocating marks, (m-1), (m-2), ... ,0, to the corresponding alternatives. All the marks received by an alternative from the members are summed to form a Borda score, and the alternative with the highest Borda score is designated the collective choice (Black, 1958). Let $B_m(A_i; N; q)$ denote the probability that A_i is the Borda winner when N group members vote on m alternatives in culture q. Only the case of three alternatives is considered as expressions for $B_m(A_i; N; q)$ become exceedingly complex when m > 3.

When $m = 3$ there are $3! = 6$ possible linear preference orderings $s_1, s_2, \ldots, s_6$ as follows:

$$s_1 : A_1 > A_2 > A_3 \qquad s_4 : A_2 > A_3 > A_1$$
$$s_2 : A_1 > A_3 > A_2 \qquad s_5 : A_3 > A_1 > A_2$$
$$s_3 : A_2 > A_1 > A_3 \qquad s_6 : A_3 > A_2 > A_1$$

where $A_i > A_j$ signifies that A_i is preferred to A_j. The probability that a voter chooses preference ordering s_h is given by q_h, where $\Sigma q_h = 1$. The set of preference ordering probabilities $q = (q_1, q_2, \ldots, q_6)$ is taken to represent the culture of which the group forms a part. The number of voters choosing preference ordering s_h is given by x_h, where $\Sigma x_h = N$.

Let a_i denote the number of voters who have A_i as first preference, b_i the number of voters who have A_i as second preference, and c_i the number of voters who have A_i as third preference. Thus,

$$a_1 = x_1 + x_2 \qquad b_1 = x_3 + x_5 \qquad c_1 = x_4 + x_6$$
$$a_2 = x_3 + x_4 \qquad b_2 = x_1 + x_6 \qquad c_2 = x_2 + x_5$$
$$a_3 = x_5 + x_6 \qquad b_3 = x_2 + x_4 \qquad c_3 = x_1 + x_3 \quad .$$

Also, let B_i represent the Borda score received by A_i. Thus, $B_i = 2a_i + b_i$. The total number of Borda marks to be distributed among the three alternatives equals $\Sigma B_i = \Sigma(2a_i + b_i) = 2N + N = 3N$.

Consider alternative A_1. For A_1 to be the Borda winner it is required that A_1 have the highest Borda score, i.e. $B_1 > B_2$ and $B_1 > B_3$. Since the total number of Borda marks equals 3N, the lowest value of B_1 with which A_1 can win is $N + 1$. The highest value of B_1 occurs when all N voters have A_1 as their first preference, i.e. $a_1 = N$ and $b_1 = c_1 = 0$. In this event $B_1 = 2N$. To obtain a lower limit for B_2 we observe that $B_1 \geq B_3 + 1$. Since $B_3 = 3N - B_1 - B_2$ it follows that $B_2 \geq 3N - 2B_1 + 1$. When $B_1 > 3N/2$ evidently $B_2 \geq 0$. Hence, for given B_1, B_2's lower limit may be written as $\max\left[3N - 2B_1 + 1, 0\right]$. The upper limit of B_2 is **clearly** $\min\left[B_1 - 1, 3N - B_1\right]$. From the definitions of a_i and b_i, $i = 1, 2, 3$, it is apparent that

$$a_1 + b_1 \leq N \quad (5.1a)$$

$$a_1 + a_2 \leq N \quad (5.1b)$$

$$a_1 + a_3 \leq N \quad (5.1c)$$

$$a_2 + b_2 \leq N \quad (5.1d)$$

$$a_3 + b_3 \leq N \quad (5.1e)$$

$$b_1 + b_2 \leq N \quad (5.1f)$$

For given B_1 and B_2, we have from inequality (5.1a) : $a_1 + B_1 - 2a_1 \leq N$, or $a_1 \geq B_1 - N$, thus establishing a lower bound for a_1. It will be observed that a_1's lower bound depends only on B_1. Now, inequalities (5.1b) and (5.1d) may simultaneously become equalities when $B_1 > 4N/3$ (determined by setting $a_2 = N - a_1$, $b_2 = a_1$, and $b_1 = 0$). It follows that $a_1 + 2a_2 + b_2 \leq 2N$, or $a_1 \leq 2N - B_2$. Similarly, from (5.1c) and (5.1e), it follows that $a_1 \leq 2N - B_3$, or $a_1 \leq B_1 + B_2 - N$. On the other hand, when $B_1 \leq 4N/3$, a_1 is limited only by B_1, that is $a_1 \leq B_1/2$. Thus, the upper limit of a_1 equals $\min\left[B_1/2, 2N - B_2, B_1 + B_2 - N\right]$. For given B_1, B_2, and a_1, we have from inequality (5.1f) : $B_1 - 2a_1 + B_2 - 2a_2 \leq N$, or $2a_2 \geq B_1 + B_2 - 2a_1 - N$, or $a_2 \geq \left\{(B_1 + B_2 - 2a_1 - N + 1)/2\right\}$ where the special brackets signify that the integer value of the expression within is required. Also, from inequality (5.1d) we have $a_2 + B_2 - 2a_2 \leq N$, or $a_2 \geq B_2 - N$. If $B_2 < N$, clearly $a_2 \geq 0$. Hence, the lower limit of a_2 equals $\max[0, B_2 - N, (B_1 + B_2 - 2a_1 - N + 1)/2]$. With regard to a_2's upper limit, evidently $a_2 \leq B_2/2$ and also $a_2 \leq N - a_1$. Further, since $a_1 + a_2 + a_3 = N$ it follows from inequality (5.1e) that $a_1 + a_2 + 2a_3 + b_3 \leq 2N$, or $a_2 \leq 2N - B_3 - a_1$, or $a_2 \leq B_1 + B_2 - N - a_1$. Therefore, the upper limit of a_2 equals $\min[B_2/2, N - a_1, B_1 + B_2 - N - a_1]$.

For given values of B_1, B_2, a_1, and a_2, all of the a_i, b_i, and c_i, $i = 1, 2, 3$, are determined. Further, knowledge of the value of one of the x_j also leads to the determination of all of the x_j, $j = 1, 2, \ldots, 6$. Consider x_1. Now, the a_i, b_i, and c_i, $i = 1, 2, 3$, constitute the nine possible pairwise summations which result when each member of the set (x_1, x_4, x_5) is added in turn to each member of the set (x_2, x_3, x_6). Therefore, for given a_i, b_i, and c_i, $i = 1, 2, 3$, if all members of one set alter their values in the same direction and by the same amount, and all members of the other set alter their values in the opposite direction by exactly the same amount, the values of a_i, b_i, and c_i, $i = 1, 2, 3$, remain unchanged. It follows that the smallest of x_2, x_3, and x_6 may equal zero. Since $a_1 = x_1 + x_2$, $b_2 = x_1 + x_6$, and $c_3 = x_1 + x_3$, clearly x_1 has an upper limit equal to $\min[a_1, b_2, c_3]$. Similarly, if x_1 is the smallest of x_1, x_4, and x_5 then x_1's lower limit equals zero; if x_4 is the smallest of the three then x_1's lower limit equals $x_1 - x_4$, or $a_1 - b_3$; and if x_5 is smallest then x_1's lower limit equals $x_1 - x_5$, or $a_1 - c_2$. Hence, x_1 has a lower limit equal to $\max[0, a_1 - b_3, a_1 - c_2]$.

In the context of the multinomial social choice model the above results yield the following expression for the probability that A_1 is the Borda winner

$$B_3(A_1;\ N;\ q) = \sum^{5} \frac{N!}{x_1!x_2!\cdots\cdots x_6!}\, q_1^{x_1}\, q_2^{x_2} \cdots\cdots q_6^{x_6} \tag{5.2}$$

where $\sum^{5}$ is a 5-fold summation whose variables of summation have the following limits

$$N+1 \leq B_1 \leq 2N$$

$$\max\left[3N-2B_1+1,0\right] \leq B_2 \leq \min\left[B_1-1,3N-B_1\right]$$

$$B_1-N \leq a_1 \leq \min\left[B_1/2,2N-B_2,B_1+B_2-N\right] \tag{5.3}$$

$$\max\left[\left\{(B_1+B_2-2a_1-N+1)/2\right\},B_2-N,0\right] \leq a_2 \leq \min\left[B_2/2,N-a_1,B_1+B_2-N-a_1\right]$$

$$\max\left[a_1-b_3,a_1-c_2,0\right] \leq x_1 \leq \min\left[a_1,b_2,c_3\right],$$

and where $x_2 = a_1 - x_1$, $x_3 = c_3 - x_1$, $x_4 = a_2 - x_3$, $x_5 = b_1 - x_3$, and $x_6 = b_2 - x_1$. Similar expressions may be obtained for $B_3(A_2;\ N;\ q)$ and $B_3(A_3;\ N;\ q)$.

5.2 BORDA INDECISION

Let $BI_m(N;\ q)$ represent the probability of Borda indecision when N group members are voting on m alternatives in culture q. As $BI_m(N;\ q)$ is the likelihood that a single winning alternative does not emerge, we may write

$$BI_m(N;\ q) = 1 - \sum_{i=1}^{m} B_m(A_i;\ N;\ q) \tag{5.4}$$

Alternatively, if $BT_m(A_i,\ A_j;\ N;\ q)$ denotes the probability that only A_i and A_j tie with the highest Borda score, $BT_m(A_i,\ A_j,\ A_k;\ N;\ q)$ the probability that only A_i, A_j, and A_k all tie with the highest Borda score, and so on, then we have

$$BI_m(N;\ q) = \sum_{i<j} BT_m(A_i, A_j;\ N;\ q) + \sum_{i<j<k} BT_m(A_i, A_j, A_k;\ N;\ q)$$

$$+ \ldots\ldots + BT_m(A_1,\ A_2,\ \ldots,\ A_m;\ N;\ q) \qquad (5.5)$$

With regard to equation (5.4) an expression for $B_m(A_i;\ N;\ q)$ was derived in section 5.1. Nevertheless, equation (5.5) provides a solution which is much quicker to compute, because considerably fewer distributions of the voters over the preference orderings result in a tie than in a win.

Once again, only the case of three alternatives is considered because of the complexity of solutions when $m > 3$. Accordingly, expressions are developed for $BT_3(A_i,\ A_j;\ N;\ q)$ and $BT_3(A_1,\ A_2,\ A_3;\ N;\ q)$.

Let B represent the highest Borda score received by an alternative. If two alternatives tie for first place then each has a Borda score of B. Since the total number of Borda marks to be distributed among three alternatives equals 3N, in the event of a pairwise tie the lower bound of B is $N + 1$ and the upper bound of B is $\{3N/2\}$, where $\{x\}$ denotes the integer value of x.

Consider alternatives A_1 and A_2. In the event of a winners' tie $B = B_1 = B_2 = 2a_1 + b_1 = 2a_2 + b_2$. From the same considerations as in section 5.1 we obtain a lower bound for a_1 equal to $B - N$, and an upper bound for a_1 equal to $\min[B/2,\ 2N-B]$. Note that a_1's upper bound is simpler than in (5.3) because $B_1 + B_2 - N = 2N - B_3$ is always larger than $2N - B$ since $B > B_3$. Similarly, a_2's lower bound becomes $\max[B - a_1 - k,\ B - N]$, where $2k + 1 = N$ if N is odd and $2k = N$ if N is even, while a_2's upper bound becomes $\min[B/2,\ N - a_1]$. The limits of x_1 remain unchanged.

By the multinomial social choice model the probability that

A_1 and A_2 are involved in a Borda winners' tie is

$$BT_3(A_1, A_2; N; q) = \overset{4}{\Sigma} \frac{N!}{x_1!x_2!\ldots\ldots x_6!} q_1^{x_1} q_2^{x_2} \ldots\ldots q_6^{x_6} \qquad (5.6)$$

where $\overset{4}{\Sigma}$ is a 4-fold summation whose variables of summation have the following limits

$$\begin{aligned} N+1 &\leq B \leq \left\{3N/2\right\} \\ B-N &\leq a_1 \leq \min\left[B/2,\ 2N-B\right] \qquad (5.7) \\ \max\left[B-a_1-k,\ B-N\right] &\leq a_2 \leq \min\left[B/2,\ N-a_1\right] \\ \max\left[a_1-b_3,\ a_1-c_2,\ 0\right] &\leq x_1 \leq \min\left[a_1,\ b_2,\ c_3\right] \end{aligned}$$

and where $x_2 = a_1 - x_1$, $x_3 = c_3 - x_1$, $x_4 = a_2 - x_3$, $x_5 = b_1 - x_3$, and $x_6 = b_2 - x_1$. Similar expressions may be obtained for $BT_3(A_1, A_3; N; q)$ and $BT_3(A_2, A_3; N; q)$.

When all three alternatives tie with a Borda score of B then clearly B = N. By the same line of reasoning as in the previous section the upper bounds of a_1 and a_2 both equal $\left\{B/2\right\}$, or k, where 2k + 1 = N if N is odd and 2k = N if N is even. The other expressions involved in the upper limits of a_1 and a_2 given by (5.3) become redundant when $B_1 = B_2 = B_3 = N$. The lower bound of a_1 is reached when both a_2 and a_3 attain their maximum values, i.e. when $a_2 = a_3 = k$, in which event $a_1 = N - 2k$. For a given value of a_1, a_2 reaches its lowest value when a_3 is at its highest value. Thus, a_2's lower limit equals $N - k - a_1$. The limits of x_1 remain as they were in section 5.1. Hence, the probability that A_1, A_2, and A_3 are involved in a Borda winners' tie is

$$BT_3(A_1,A_2,A_3; N; q) = \overset{3}{\Sigma} \frac{N!}{x_1!x_2!\ldots\ldots x_6!} q_1^{x_1} q_2^{x_2} \ldots\ldots q_6^{x_6} \qquad (5.8)$$

where $\overset{3}{\Sigma}$ is a 3-fold summation whose variables of summation have the following limits

$$N-2k \leq a_1 \leq k$$
$$N-k-a_1 \leq a_2 \leq k \qquad (5.9)$$
$$\max\left[a_1-b_3,\ a_1-c_2,\ 0\right] \leq x_1 \leq \min\left[a_1,\ b_2,\ c_3\right]$$

and where $x_2 = a_1 - x_1$, $x_3 = c_3 - x_1$, $x_4 = a_2 - x_3$, $x_5 = b_1 - x_3$, and $x_6 = b_2 - x_1$.

Finally, by equations (5.5), (5.6), and (5.8) we have an expression for the probability of Borda indecision :

$$\begin{aligned} BI_3(N;\ q) &= BT_3(A_1,\ A_2;\ N;\ q) + BT_3(A_1,\ A_3;\ N;\ q) \\ &\quad + BT_3(A_2,\ A_3;\ N;\ q) + BT_3(A_1,\ A_2,\ A_3;\ N;\ q) \\ &= \overset{4}{\Sigma} \frac{N!}{x_1!x_2!x_3!x_4!x_5!x_6!} \Big[q_1^{x_1} q_2^{x_2} q_3^{x_3} q_4^{x_4} q_5^{x_5} q_6^{x_6} \\ &\qquad + q_1^{x_2} q_2^{x_1} q_3^{x_5} q_4^{x_6} q_5^{x_3} q_6^{x_4} \\ &\qquad + q_1^{x_6} q_2^{x_5} q_3^{x_4} q_4^{x_3} q_5^{x_2} q_6^{x_1} \Big] \\ &\quad + \overset{3}{\Sigma} \frac{N!}{x_1!x_2!,\ldots,x_6!} q_1^{x_1} q_2^{x_2},\ldots,q_6^{x_6} \ . \end{aligned} \qquad (5.10)$$

where $\overset{4}{\Sigma}$ is as defined in (5.7) and $\overset{3}{\Sigma}$ is as defined in (5.9). It will be observed that the three pairwise tie probabilities have been brought together under a single 4-fold summation sign whose variables of summation are given, along with their limits, by (5.7). Of course, each distribution of the values $x_1, x_2, \ldots, x_6$, determined by the variables of summation in (5.7), gives rise to a tie between A_1 and A_2. However, for every such distribution a corresponding one, which results in a tie between A_1 and A_3, may be found by interchanging the labels A_2 and A_3 in all the preference orderings. Thus, x_1 votes are now received by preference ordering s_2, x_2 votes by s_1, x_3 votes by s_5, x_4 votes by s_6, x_5 votes by s_3, and x_6 votes by s_4. Similarly, corresponding distributions producing a tie between

A_2 and A_3 may be obtained from the first distributions by interchanging the labels A_1 and A_3, so that x_1 votes are now received by preference ordering s_6, x_2 votes by s_5, x_3 votes by s_4, x_4 votes by s_3, x_5 votes by s_2, and x_6 votes by s_1. It follows that knowledge of the limits of the variables of summation for only one of the three pairwise tie probabilities is sufficient to permit the evaluation of all three.

5.3 EFFECT OF GROUP SIZE, NUMBER OF ALTERNATIVES, AND CULTURE ON THE LIKELIHOOD OF BORDA INDECISION

By means of the formulae derived in section 5.2 the likelihood of Borda indecision in case $m = 3$ was calculated in each of six cultures for group sizes $N = 3, 4, \ldots, 100$. Table 5.1 contains the resultant probability values. Culture (i) is the familiar equiprobable culture; cultures (iii) and (iv) have the property of single-peakedness discussed in chapter 1; culture (v) is the culture in which the likelihood of the paradox of voting is at its maximum; and cultures (ii) and (vi) represent minor departures from cultures (i) and (v), respectively.

In all cultures the likelihood of Borda indecision diminishes towards zero as N increases. In some cultures the approach toward the asymptote is more prolonged than in others. In particular, if in a given culture two or more alternatives receive exactly the same expected number of marks from a single voter, and if this expected number of marks is not less than unity, then the likelihood of Borda indecision in that culture will tend to approach the asymptote at a somewhat leisurely pace. Cultures (i), (iii) and (v) are of this type. For example, in culture (v) formula (5.10) simplifies to

A_2 and A_3 may be obtained from the first distributions by interchanging the labels A_1 and A_3, so that x_1 votes are now received by preference ordering s_6, x_2 votes by s_5, x_3 votes by s_4, x_4 votes by s_3, x_5 votes by s_2, and x_6 votes by s_1. It follows that knowledge of the limits of the variables of summation for only one of the three pairwise tie probabilities is sufficient to permit the evaluation of all three.

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$$BI_3(N;q) = (3/2)\cdot\frac{N!}{(2N/3)!(N/3)!}(2/3)^{2N/3}(1/3)^{N/3}$$

$$-(1/2)\cdot\frac{N!}{(N/3)!(N/3)!(N/3)!}(1/3)^{N} \qquad (5.11)$$

if N is a multiple of 3;

= 0 otherwise .

When N = 9999 the likelihood of Borda indecision is still quite high at .01265 in this culture.

However, the asymptotic behaviour of cultures such as (i), (iii) and (v) is rather unrepresentative. Exact equality of two or more alternatives in terms of their expected number of marks is an artificial circumstance which is unlikely to be encountered in practice. Moreoever, minor deviations from equality produce a more rapid approach to the asymptote. Thus, in cultures (ii) and (vi) which are very similar to cultures (i) and (v) respectively, except that no two alternatives have exactly the same expected number of marks, the likelihood of Borda indecision decreases towards zero at a much quicker rate.

In some cultures the likelihood of Borda indecision varies with N in a dramatically non-monotonic manner. Thus, in culture (v) the probability of indecision is substantial when N is a multiple of 3, and equals zero when N is not a multiple of 3. In other cultures, e.g. (i), (ii) and (iv), the probability of indecision displays few fluctuations and generally decreases monotonically with N.

The upper limit of the likelihood of Borda indecision appears to be provided by different cultures depending on the value of N. When N is a multiple of 3, culture (v) would seem to represent the upper limit. When N is even and not a multiple of 3, culture (iii) seems to constitute the upper limit. When N is odd and not a multiple of 3 no single culture provides

$$BI_3(N; q) = (3/2) \cdot \frac{N!}{(2N/3)!(N/3)!} (2/3)^{2N/3} (1/3)^{N/3}$$

$$- (1/2) \cdot \frac{N!}{(N/3)!(N/3)!(N/3)!} (1/3)^{N} \qquad (5.11)$$

if N is a multiple of 3;

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TABLE 5.1

PROBABILITY OF BORDA INDECISION FOR N GROUP MEMBERS AND m = 3 ALTERNATIVES IN EACH OF SIX CULTURES :

(i) q = (1/6, 1/6, 1/6, 1/6, 1/6, 1/6),

(ii) q = (1/4, 1/6, 1/6, 1/6, 1/6, 1/12),

(iii) q = (1/2, 0, 1/4, 0, 0, 1/4), ... continued overleaf

N	Culture (i)	Culture (ii)	Culture (iii)
3	.13889	.14583	.04688
4	.19444	.17622	.37500
5	.14660	.14628	.03906
6	.15111	.13603	.31616
7	.13503	.12763	.02563
8	.13098	.11740	.27856
9	.12311	.11163	.01570
10	.11843	.10454	.25090
11	.11332	.09952	.00944
12	.10928	.09439	.22939
13	.10543	.09011	.00568
14	.10209	.08608	.21223
15	.09899	.08247	.00343
16	.09620	.07914	.19828
17	.09362	.07608	.00209
18	.09125	.07325	.18674
19	.08905	.07062	.00128
20	.08701	.06817	.17703
49	.05743	.03264	.00000
50	.05689	.03201	.11228
99	.04106	.01485	.00000
100	.04086	.01466	.07959

TABLE 5.1 (continued)

(iv) q = (1/4, 0, 1/4, 1/4, 0, 1/4),

(v) q = (1/3, 0, 0, 1/3, 1/3, 0), and

(vi) q = (1/4, 1/12, 0, 1/3, 1/3, 0).

N	Culture (iv)	Culture (v)	Culture (vi)
3	.09375	.55556	.42361
4	.16406	.00000	.01852
5	.07813	.00000	.14596
6	.08057	.43210	.25428
7	.05127	.00000	.04013
8	.04337	.00000	.14608
9	.03140	.36702	.17245
10	.02475	.00000	.05691
11	.01888	.00000	.13469
12	.01461	.32507	.12817
13	.01135	.00000	.06768
14	.00878	.00000	.12046
15	.00687	.29509	.10309
16	.00533	.00000	.07334
17	.00418	.00000	.10688
18	.00327	.27226	.08838
19	.00256	.00000	.07527
20	.00201	.00000	.09520
49	.00000	.00000	.05128
50	.00000	.00000	.05041
99	.00000	.12306	.03145
100	.00000	.00000	.03120

the upper limit for all N. However, a culture which comes fairly close to fulfilling this function is culture (i), the equiprobable culture. In this culture, for all odd N not a multiple of three, the likelihood of Borda indecision is very near its maximum, except in cases N = 5 and N = 11 when the likelihood is lower than its upper limit by around .06 and .02 respectively.

In general, for very large groups (N > 10000) the upper limit of the likelihood of Borda indecision would appear to be approximately .01 (culture (v)), with the actual likelihood in almost all cultures being considerably less than this value. Moderately large groups, of around 100 members, may have as much as a one-in-eight chance of an equivocal outcome (culture (v)). Again, in most cultures the actual likelihood of indecision is much smaller than this upper limit. When group size is small ($N \leq 20$) the upper limit of the likelihood of Borda indecision varies considerably over N. Table 5.2 contains the approximate upper limits of the likelihood of Borda indecision for $N \leq 20$.

TABLE 5.2

APPROXIMATE UPPER LIMIT OF THE LIKELIHOOD OF BORDA INDECISION FOR N GROUP MEMBERS AND m = 3 ALTERNATIVES.

N	UPPER LIMIT	N	UPPER LIMIT
3	.56	12	.33
4	.38	13	.11
5	.21	14	.21
6	.43	15	.30
7	.14	16	.20
8	.28	17	.11
9	.37	18	.27
10	.25	19	.09
11	.14	20	.18

If a decision making body, using the Borda procedure to decide among three alternatives, wishes to adopt a group size which will minimise the maximum likelihood of indecision, then the values in Table 5.2 indicate the best means of achieving this goal. For example, if it is required that the group consist of less than ten members then group size $N = 7$ offers the lowest upper limit for the likelihood of indecision. If the group has to comprise fewer than 20 members then group sizes $N = 13$, $N = 17$, and $N = 19$ provide the lowest likelihood of indecision should the culture most productive of indecision prevail.

With groups of less than 50 members certain cultures would appear to exhibit greater stability than others under the Borda procedure. In a stable culture minor changes in the preference ordering probabilities q_i, $i = 1, 2, ..., 6$, have little effect on the likelihood of Borda indecision, while changes of a similar magnitude in the q_i, $i = 1, 2, ..., 6$, of an unstable culture may produce a substantial alteration in the likelihood of Borda indecision. Culture (i) is of the former type while culture (v) is of the latter type. Thus, culture (ii) differs slightly from culture (i) but yields essentially the same likelihood of indecision when $N < 50$. By contrast, culture (vi) differs from culture (v) to a similar modest degree but can produce a quite dissimilar likelihood of indecision. For example, when $N = 8$ the likelihood of Borda indecision equals zero in culture (v) and .14608 in culture (vi).

Our discussion so far has centred on the case of three alternatives. When $m > 3$ the development of an expression along the lines of formula (5.10) is a lengthy, cumbersome operation. Accordingly, a different procedure is adopted in case $m = 4$. (Limitations of computer time preclude consideration of cases where $m > 4$.) A computer search algorithm is constructed to find those distributions of the voters over the preference orderings which produce Borda indecision. Feeding the results into the multinomial social choice model, the likelihood of Borda indecision in case $m = 4$ is obtained. The method will not be further outlined as it is essentially the same as that used by Garman and Kamien (1968) and Niemi and Weisberg (1968) to calculate the probability of the paradox of voting. Although straightforward to program, this approach requires a vast amount of computer time. Hence, only small group sizes ($N \leq 20$) are considered.

When $m = 4$ there are $4! = 24$ possible linear preference orderings $t_1, t_2, ..., t_{24}$, as outlined in chapter 2. The probability that a voter chooses preference ordering t_i is denoted by r_i. Table 5.3 provides the likelihood of Borda indecision in each of three cultures when $N \leq 20$. Culture (i) in which $r_i = 1/24$, all i, is the equiprobable culture; culture (ii) in which $r_1 = r_{24} = 1/4$, $r_7 = r_9 = 1/8$, $r_{15} = r_{16} = r_{18} = 1/12$, $r_i = 0$, all other i, is single-peaked; and culture (iii) in which $r_1 = r_{10} = r_{17} = r_{19} = 1/4$, $r_i = 0$, all other i, is the culture which produces the maximum likelihood of the paradox of voting.

In many respects the likelihood of Borda indecision when $m = 4$ conducts itself in the same manner as when $m = 3$. As N increases the probability of indecision shrinks towards zero in all cultures. Also, the equiprobable culture is more stable than the culture in which the likelihood of the paradox of voting is at a maximum. The main difference is that for given N the upper limit of the likelihood of Borda indecision is higher when $m = 4$.

TABLE 5.3

PROBABILITY OF BORDA INDECISION FOR N GROUP MEMBERS AND m = 4 ALTERNATIVES IN EACH OF THREE CULTURES :

(i) $r_i = 1/24$, all i ;

(ii) $r_1 = r_{24} = 1/4$, $r_4 = r_9 = 1/8$, $r_{15} = r_{16} = r_{18} = 1/12$, $r_i = 0$, all other i ;

(iii) $r_1 = r_{10} = r_{17} = r_{19} = 1/4$, $r_i = 0$, all other i.

N	Culture (i)	Culture (ii)	Culture (iii)
3	.18403	.13932	.00000
4	.18309	.26237	.57813
5	.16406	.15801	.00000
6	.14960	.18318	.15625
7	.14201	.14242	.00000
8	*	.14115	.45587
9	*	.12271	.00000
10	*	.11636	.12305
11	*	.10638	.00000
12	*	.10038	.39011
13	*	.09409	.00000
14	*	.08934	.10474
15	*	.08497	.00000
16	*	.08132	.34721
17	*	.07807	.00000
18	*	.07524	.09274
19	*	.07272	.00000
20	*	.07047	.31629

* Calculation of these values requires a prohibitively large amount of computer time.

CHAPTER 6

THE COMPARATIVE LIKELIHOOD OF PLURALITY, CONDORCET, AND BORDA INDECISION

6.1 SINGLE PROCEDURES

The case of $m = 3$ alternatives only is considered.

In the context of the plurality procedure a culture is defined in terms of the first preference probabilities p_1, p_2, and p_3. Since $p_1 = q_1 + q_2$, $p_2 = q_3 + q_4$, and $p_3 = q_5 + q_6$, it follows that for every plurality culture there is an infinite number of Borda or Condorcet cultures.

Table 6.1 incorporates the likelihoods of plurality, Condorcet, and Borda indecision for group sizes $N = 3, 4, \ldots, 100$ in each of five cultures. The cultures examined include equiprobability (culture (i)), value restriction (cultures (ii), (iii) and (iv)), and maximum likelihood of the paradox of voting (culture (v)). The information provided in Table 6.1 together with analytic considerations forms the basis for a comparison of the three social choice procedures. Each procedure is evaluated in relation to every other one in terms of its propensity to indecision.

<u>Plurality and Condorcet</u>. When N = 3 and N = 5 it can be shown that the likelihood of Condorcet indecision is always less than or equal to that of plurality indecision across all cultures. On the other hand, when N = 4 and N = 6 it can be demonstrated that the likelihood of plurality indecision is always less than or equal to that of Condorcet indecision across all cultures. Since the method of proof is essentially the same in all four cases we need consider only one case, say N = 4.

By formula (3.1) we obtain, after simplification, the following expression for the probability that a plurality winner exists when N = 4:

$$
\begin{aligned}
p(\text{plurality winner exists}) &= \sum_{i=1}^{3} P_3(A_i;\ 4;\ p) \\
&= p_1^4 + 4p_1^3(p_2 + p_3) + 12p_1^2\, p_2\, p_3 \\
&\quad + p_2^4 + 4p_2^3(p_1 + p_3) + 12p_2^2\, p_1\, p_3 \\
&\quad + p_3^4 + 4p_3^3(p_1 + p_2) + 12p_3^2\, p_1\, p_2 \qquad (6.1)
\end{aligned}
$$

By formula (4.10) the likelihood that a Condorcet winner exists when N = 4 is given by

$$
\begin{aligned}
p(\text{Condorcet winner exists}) &= \sum_{i=1}^{3} C_3(A_i;\ 4;\ q) \\
&= p_1^4 + 4p_1^3(p_2 + p_3) + 12p_1^2\, q_3\, q_5 \\
&\quad + p_2^4 + 4p_2^3(p_1 + p_3) + 12p_2^2\, q_1\, q_6 \\
&\quad + p_3^4 + 4p_3^3(p_1 + p_2) + 12p_3^2\, q_2\, q_4 \qquad (6.2)
\end{aligned}
$$

where $p_1 = q_1 + q_2$, $p_2 = q_3 + q_4$, and $p_3 = q_5 + q_6$. On comparing expressions (6.1) and (6.2) it is evident that the former is at least as great as the latter. That is, a plurality winner is at least as likely to emerge as a Condorcet winner. Hence, when N = 4 the probability of plurality indecision is less than or equal to the probability of Condorcet indecision across all cultures.

When N > 6 we find from Table 6.1 that in culture (iii) the likelihood of Condorcet indecision is less than that of plurality indecision for all N. By contrast, in culture (v) the likelihood of Condorcet indecision is greater than that of plurality indecision for all N > 6. Therefore, when N > 6 neither procedure is consistently more likely to be decisive than the other across all cultures.

As was established in chapter 3, the upper limit of the likelihood of plurality indecision occurs in culture $p = (1/3, 1/3, 1/3)$ for all odd N and in culture $p = (1/2, 1/2, 0)$ for almost all even N. The upper limit of the likelihood of Condorcet indecision was earlier found to occur in culture (v). A comparison of these cultures clearly illustrates that when N > 6 the upper limit of the likelihood of Condorcet indecision is much higher than that of plurality indecision.

To summarise, when N > 6 neither procedure is consistently more likely to be decisive than the other across all cultures, though the upper limit of the probability of Condorcet indecision is considerably higher than the upper limit of the probability of plurality indecision. When N = 3 and N = 5 the likelihood of Condorcet indecision is less than or equal to the likelihood of plurality indecision across all cultures. When N = 4 and N = 6 the reverse is true.

Plurality and Borda. In culture (iii) the likelihood of Borda indecision is less than that of plurality indecision for all N. On the other hand, in culture (ii) when N is even and in culture (iv) when N is odd, the likelihood of plurality indecision is less than that of Borda indecision for all N. Hence, there is no value of N for which one procedure is consistently more likely to be decisive than the other across all cultures.

The upper limit of the likelihood of Borda indecision was previously found to be provided, approximately, by culture (i) when N is odd and not a multiple of 3, by culture (ii) when N is even and not a multiple of 3, and by culture (v) when N is a multiple of 3. The upper limit of the likelihood of plurality indecision was earlier ascertained to occur in culture $p = (1/3, 1/3, 1/3)$ for all odd N and in culture $p = (1/2, 1/2, 0)$ for almost all even N. A comparison of these cultures reveals that when N is odd and not a multiple of 3 the upper limit of the likelihood of plurality indecision is greater than that of Borda indecision. When N is even and not a multiple of 3 the two upper limits differ only marginally when $N < 20$ and are identical when $N > 50$. In effect, the Borda upper limit in this case approaches quickly (by $N = 50$) an expression identical to that of the plurality upper limit, namely

$$\frac{N!}{(N/2)!\,(N/2)!}\,(1/2)^N. \tag{6.3}$$

Finally, when N is a multiple of 3 the upper limit of the likelihood of Borda indecision is greater than that of plurality indecision.

Condorcet and Borda. When N = 3 it can be shown that the likelihood of Condorcet indecision is less than or equal to that of Borda indecision across all cultures. The probability that a Condorcet winner exists may be obtained by means of formula (4.10). Thus, when N = 3.

$$\begin{aligned} p(\text{Condorcet winner exists}) &= \sum_{i=1}^{3} C_3(A_i;\ 3;\ q) \\ &= p_1^3 + 3p_1^2(p_2 + p_3) + 6p_1\ q_3\ q_5 \\ &\quad + p_2^3 + 3p_2^2(p_1 + p_3) + 6p_2\ q_1\ q_6 \\ &\quad + p_3^3 + 3p_3^2(p_1 + p_2) + 6p_3\ q_2\ q_4. \end{aligned} \qquad (6.4)$$

The probability that a Borda winner exists when N = 3 may be found, after some manipulation, through formula (5.2), giving

$$\begin{aligned} p(\text{Borda winner exists}) &= \sum_{i=1}^{3} B_3(A_i;\ 3;\ q) \\ &= p_1^3 + 3p_1^2(p_2 + p_3) + 6p_1\ q_3\ q_5 \\ &\quad + p_2^3 + 3p_2^2(p_1 + p_3) + 6p_2\ q_1\ q_6 \\ &\quad + p_3^3 + 3p_3^2(p_1 + p_2) + 6p_3\ q_2\ q_4 \\ &\quad - 3q_1^2\ q_4 - 3q_2^2\ q_6 - 3q_3^2\ q_2 \\ &\quad - 3q_4^2\ q_5 - 3q_5^2\ q_1 - 3q_6^2\ q_3\ . \end{aligned} \qquad (6.5)$$

Since expression (6.4) is at least as great as expression (6.5), a Condorcet winner is at least as likely to emerge as a Borda winner. Therefore, when N = 3 the probability of Condorcet indecision is less than or equal to the probability of Borda indecision across all cultures.

On the other hand, when N = 4 it can be demonstrated by similar means that the likelihood of Borda indecision is less than or equal to the likelihood of Condorcet indecision across all cultures.

When N > 4 we find that in culture (v) the likelihood of Borda indecision is less than that of Condorcet indecision for all N. In culture (iii) when N is odd the likelihood of Condorcet indecision is less than that of Borda indecision. Therefore, for all odd N > 4 neither procedure is consistently more likely to be decisive than the other across all cultures. When N is even the situation is slightly more complex. In culture q = (0, 1/3, 2/3, 0, 0, 0) it is evident that the probability of Condorcet indecision is

$$CI_3(N; q) = \frac{N!}{(N/2)!\,(N/2)!} (1/3)^{N/2} (2/3)^{N/2} \quad N \text{ even};$$

$$= 0 \quad N \text{ odd}. \tag{6.6}$$

Also, the probability of Borda indecision is

$$BI_3(N; q) = \frac{N!}{(N/3)!\,(2N/3)!} (1/3)^{N/3} (2/3)^{2N/3} \quad N \text{ a multiple of } 3;$$

$$= 0 \quad \text{otherwise}. \tag{6.7}$$

When N is even and a multiple of 3 the former expression is clearly smaller than the latter. Hence, the likelihood of Condorcet indecision is less than that of Borda indecision. Consideration of cultures intermediate between culture q = (0, 1/3, 2/3, 0, 0, 0) and culture (iv) reveals that it is possible for the likelihood of Condorcet indecision to be less than that of Borda indecision for all even values of N > 4 except N = 8, 10, 14 and 16. This is not an exact result because there do exist cultures in which, when N = 8, 10, 14 or 16, the likelihood of Condorcet indecision is actually less than that of Borda indecision, e.g. q = (.49, .49, 0, .01, 0, .01). However, the difference between the two likelihoods in such cultures is of a very low order of magnitude. Hence, it may be

TABLE 6.1

LIKELIHOOD OF PLURALITY, CONDORCET, AND BORDA INDECISION FOR N GROUP MEMBERS AND m = 3 ALTERNATIVES IN EACH OF FIVE CULTURES.

CULTURE (i) q = (1/6, 1/6, 1/6, 1/6, 1/6, 1/6)

N	PLURALITY	CONDORCET	BORDA
3	.22222	.05556	.13889
4	.22222	.55556	.19444
5	.37037	.06944	.14660
6	.20576	.49126	.15111
7	.19204	.07502	.13503
8	.28807	.44810	.13098
9	.18137	.07798	.12311
10	.17284	.41660	.11843
11	.24254	.07981	.11332
12	.16430	.39230	.10928
13	.15822	.08107	.10543
14	.21311	.37280	.10209
15	.15140	.08198	.09899
16	.14680	.35670	.09620
17	.19212	.08267	.09362
18	.14122	.34311	.09125
19	.13758	.08322	.08905
20	.17621	.33143	.08701
49	.09306	.08601	.05743
50	.10879	.24779	.05689
99	.06798	.08688	.04106
100	.06764	.20292	.04086

TABLE 6.1 (continued)

CULTURE (ii) : q = (1/2, 0, 1/4, 0, 0, 1/4)

N	PLURALITY	CONDORCET	BORDA
3	.18750	.00000	.04688
4	.21094	.49219	.37500
5	.29297	.00000	.03906
6	.17090	.40527	.31616
7	.15381	.00000	.02563
8	.20615	.34286	.27856
9	.13298	.00000	.01570
10	.12377	.29680	.25090
11	.15377	.00000	.00944
12	.10756	.26221	.22939
13	.10059	.00000	.00568
14	.11981	.23580	.21223
15	.08817	.00000	.00343
16	.08285	.21527	.19828
17	.09566	.00000	.00209
18	.07305	.19903	.18674
19	.06885	.00000	.00128
20	.07764	.18595	.17703
49	.01344	.00000	.00000
50	.01371	.11236	.11228
99	.00115	.00000	.00000
100	.00110	.07959	.07959

TABLE 6.1 (continued)

CULTURE (iii) : q = (1/4, 0, 1/4, 1/4, 0, 1/4)

N	PLURALITY	CONDORCET	BORDA
3	.18750	.00000	.09375
4	.21094	.39844	.16406
5	.29297	.00000	.07813
6	.17090	.25879	.08057
7	.15381	.00000	.05127
8	.20615	.17197	.04337
9	.13298	.00000	.03140
10	.12377	.11656	.02475
11	.15377	.00000	.01888
12	.10756	.08024	.01461
13	.10059	.00000	.01135
14	.11981	.05591	.00878
15	.08817	.00000	.00687
16	.08285	.03932	.00533
17	.09566	.00000	.00418
18	.07305	.02785	.00327
19	.06885	.00000	.00256
20	.07764	.01984	.00201
49	.01344	.00000	.00000
50	.01371	.00017	.00000
99	.00115	.00000	.00000
100	.00110	.00000	.00000

TABLE 6.1 (continued)

CULTURE (iv) : q = (.1, .2, .6, 0, .1, 0)

N	PLURALITY	CONDORCET	BORDA
3	.10800	.00000	.21900
4	.22140	.37260	.10320
5	.14580	.00000	.17440
6	.15066	.28674	.11826
7	.09299	.00000	.10954
8	.12344	.23593	.12100
9	.07869	.00000	.09432
10	.08548	.20195	.10097
11	.07125	.00000	.09371
12	.06785	.17703	.08721
13	.05654	.00000	.08765
14	.05714	.15756	.08200
15	.04789	.00000	.08007
16	.04602	.14172	.07805
17	.04151	.00000	.07495
18	.03866	.12846	.07353
19	.03491	.00000	.07137
20	.03311	.11715	.06956
49	.00450	.00000	.04482
50	.00423	.04046	.04438
99	.00023	.00000	.03162
100	.00021	.01034	.03146

TABLE 6.1 (continued)

CULTURE (v) : q = (1/3, 0, 0, 1/3, 1/3, 0)

N	PLURALITY	CONDORCET	BORDA
3	.22222	.22222	.55556
4	.22222	.66666	.00000
5	.37037	.37037	.00000
6	.20576	.69959	.43210
7	.19204	.48011	.00000
8	.28807	.73617	.00000
9	.18137	.56546	.36702
10	.17284	.77031	.00000
11	.24254	.63374	.00000
12	.16430	.80066	.32507
13	.15822	.68938	.00000
14	.21311	.82715	.00000
15	.15140	.73531	.29509
16	.14680	.85011	.00000
17	.19212	.77357	.00000
18	.14122	.86996	.27226
19	.13758	.80570	.00000
20	.17621	.88709	.00000
49	.09306	.97638	.00000
50	.10879	.98525	.00000
99	.06798	.99907	.12306
100	.06764	.99940	.00000

concluded that for all even N > 4 neither procedure is consistently more decisive than the other across all cultures, except when N = 8, 10, 14 or 16 in which case the Borda procedure is for all practical purposes at least as likely to be decisive as the Condorcet procedure across all cultures.

The cultures in which for given values of N the likelihoods of the two procedures reach their upper limits were previously established to be: in the case of the Condorcet procedure culture (v) for all N; and in the case of the Borda procedure culture (i) for all odd N not a multiple of 3, culture (ii) for all even N not a multiple of 3, and culture (v) for all N a multiple of 3. From a comparison of these cultures it is evident that the upper limit of the Condorcet indecision is considerably higher than that of Borda indecision for all N > 3.

To summarise, neither procedure is consistently more likely to be decisive than the other across all cultures, except when N = 3, 4, 8, 10, 14 and 16. When N = 3 the likelihood of Condorcet indecision is less than or equal to that of Borda indecision across all cultures. When N = 4, 8, 10, 14, and 16, the likelihood of Borda indecision is less than or equal to that of Condorcet indecision across all cultures. The upper limit of the likelihood of Condorcet indecision is considerably greater than that of Borda indecision for all N > 3.

6.2 AMALGAMATED PROCEDURES

Black (1958) proposed an amalgamation of the Condorcet and Borda systems of collective choice in which the Condorcet system is adopted as the primary procedure and the Borda system is brought into operation as a secondary procedure on those occasions when a Condorcet winner fails to emerge. The principle underlying Black's recommendation may be generalised in the following manner. A group should employ what it considers to be the most acceptable social choice scheme, using criteria such as those examined by Richelson (1975). In situations where the decision scheme produces an ambiguous outcome the next most acceptable procedure may be adopted. Should both primary and secondary schemes produce indecision the third most acceptable procedure may be invoked, and so on. Black's procedure is one of many possible amalgamations. For example, the order of implementation in Black's procedure might be reversed so that the Borda scheme is adopted as the primary procedure and the Condorcet scheme is invoked in the event of Borda indecision. Now, the order in which the decision schemes acceptable to a group are deployed has considerable bearing on the particular alternative to emerge as collective choice. However, for a given set of social choice schemes, each possible order of implementation results in exactly the same likelihood of indecision. Hence, for present purposes, order of implementation may be ignored.

Two amalgamated procedures are examined: the dual Condorcet/Borda amalgamation and the triple Condorcet/Borda/ plurality amalgamation. Table 6.2 contains the likelihood of indecision under both amalgamations in the same five cultures studied in Table 6.1. The values in Table 6.2 were derived by a computer search algorithm in the manner described earlier in connection with Borda indecision in case m = 4.

When N = 3 neither the dual nor the triple amalgamation is more likely to be decisive than the Condorcet procedure on its own. This is evident from a comparison of the probabilities that a winner exists under each of the three procedures. These probabilities are provided in the case of the Condorcet system by formula (6.4), in the case of the Borda system by formula (6.5), and in the case of the plurality system by

$$p(\text{plurality winner exists}) = \sum_{i=1}^{3} p_3(A_i;\ 3;\ p)$$

$$= p_1^3 + 3p_1^2(p_2 + p_3) + p_2^3 + 3p_2^2(p_1 + p_3) + p_3^3 + 3p_3^2(p_1 + p_2)\ . \qquad (6.8)$$

Expression (6.4) is at least as great as either expression (6.5) or expression (6.8), reflecting the fact that the existence of either a Borda or a plurality winner implies the existence of a Condorcet winner. Consequently, when N = 3 and the Condorcet system is adopted as the primary procedure, the dual and the triple amalgamations are redundant since they reduce to the Condorcet procedure on its own. The upper limit of the likelihood of indecision in the dual and triple amalgamations when N = 3 is therefore .22222 (culture (v)).

When N = 4 it is readily found that the existence of a Condorcet winner implies the existence of a Borda winner, giving rise to the previously established result that the likelihood of Borda indecision is always less than or equal to that of Condorcet indecision across all cultures. Hence, when N = 4 and the Borda system is adopted as the primary procedure, the dual amalgamation is redundant since it reduces to the Borda procedure on its own. The upper limit of the likelihood of indecision in the dual amalgamation when N = 4 is therefore .375 (e.g. culture (ii)). In the triple amalgamation, which by the above reasoning reduces to a Borda/plurality amalgamation when N = 4, the upper limit of the likelihood of indecision also equals .375 (e.g. when $q_1 = q_3 = 1/2$, $q_i = 0$, all other i).

When N = 5 the existence of a plurality winner is readily found to imply the existence of a Condorcet winner. Hence, when N = 5 and the plurality system is deployed as the tertiary procedure, the triple amalgamation is redundant since it reduces to the dual amalgamation on its own. The upper limit of the likelihood of indecision in the dual and triple amalgamations appears to be .0768 (e.g. when $q_1 = q_2 = q_5 = .2$, $q_4 = .4$, $q_3 = q_6 = 0$).

The dual Condorcet/Borda amalgamation formulated by Black (1958) was proposed in order to combat indecision of the cyclical majority type. Hence, as might be expected, when N > 3 a comparison of Tables 6.1 and 6.2 demonstrates that in culture (v) Black's procedure provides a substantial reduction in the likelihood of indecision which otherwise obtains under the Condorcet procedure. By contrast, in culture (ii) Black's procedure is only marginally more likely to be decisive than the Condorcet procedure. The diminished effectiveness of Black's procedure in culture (ii) occurs because the only source of indecision here is the presence of ties, a form of deadlock to which both the Condorcet and Borda schemes are susceptible. Thus, in culture (ii) the probability of Condorcet indecision, after simplification, is

$$CI_3(N;\, q) = \frac{N!}{(N/2)!\,(N/2)!}\,(1/2)^N \left[1 + (3/4)^{N/2} - (1/2)^{N/2}\right] \qquad N \text{ even};$$
$$= 0 \qquad N \text{ odd}. \qquad (6.9)$$

The expression within square brackets quickly approaches unity as N increases, so that when N > 100 the likelihood of Condorcet indecision becomes

$$CI_3(N;\ q) = \frac{N!}{(N/2)!\ (N/2)!}\,(1/2)^N \qquad N \text{ even};$$

$$= 0 \qquad N \text{ odd}. \qquad (6.10)$$

Also, in culture (ii) the probability of Borda indecision, after simplification and rearrangement, is

$$BI_3(N;\ q) = \sum_{j=k+1}^{\{2N/3\}} \frac{N!}{(2N-3j)!(3j-N)!}\,(1/2)^N\,\frac{(3j-N)!}{(2j-N)!j!}\,(1/2)^{3j-N}$$

$$\left(+\ \frac{N!}{(N/2)!\ (N/2)!}\,(1/2)^N \quad \text{if } N \text{ is even}\right) \qquad (6.11)$$

where $j = x_6$, $2k + 1 = N$ if N is odd and $2k + 2 = N$ if N is even. The expression undergoing summation in (6.11) consists of the product of two binomial density functions each with a probability of "success" equal to 1/2. Now, as j ranges from $k + 1$ to $\{2N/3\}$ the effect is that either an extreme term from one density function is multiplied by a more central term from the other, or two extreme terms are multiplied together. Therefore, as N increases the expression undergoing summation in (6.11) will decrease to zero, giving

$$BI_3(N;\ q) = \frac{N!}{(N/2)!\ (N/2)!}\,(1/2)^N \qquad N \text{ even};$$

$$= 0 \qquad N \text{ odd}. \qquad (6.12)$$

Thus, in culture (ii) as N increases $CI_3(N;\ q)$ and $BI_3(N;\ q)$ converge on the same form. By N = 100 the two likelihoods are identical. Therefore, in such a culture when N is moderately large or large, Black's procedure is no more likely to be decisive than either the Borda or the Condorcet scheme on its own.

When N is large (N > 10000) it was found previously that the

TABLE 6.2

LIKELIHOOD OF INDECISION IN TWO AMALGAMATIONS OF SOCIAL CHOICE PROCEDURES FOR N GROUP MEMBERS AND m = 3 ALTERNATIVES. THE TWO AMALGAMATIONS, (a) CONDORCET/BORDA AND (b) CONDORCET/BORDA/PLURALITY, ARE CONSIDERED IN EACH OF FIVE CULTURES :

(i) q = (1/6, 1/6, 1/6, 1/6, 1/6, 1/6),

(ii) q = (1/2, 0, 1/4, 0, 0, 1/4),

(iii) q = (1/4, 0, 1/4, 1/4, 0, 1/4),

(iv) q = (.1, .2, .6, 0, .1, 0), and

(v) q = (1/3, 0, 0, 1/3, 1/3, 0).

(a) CONDORCET/BORDA

N	(i)	(ii)	(iii)	(iv)	(v)
3	.05556	.00000	.00000	.00000	.22222
4	.19444	.37500	.16406	.10320	.00000
5	.04630	.00000	.00000	.00000	.00000
6	.14918	.31250	.07324	.04048	.37037
7	.03601	.00000	.00000	.00000	.00000
8	.12578	.27344	.03311	.01648	.00000
9	.02959	.00000	.00000	.00000	.27740
10	.10999	.24609	.01514	.00690	.00000
11	.02571	.00000	.00000	.00000	.00000
12	.09809	.22559	.00699	.00296	.29992
13	.02321	.00000	.00000	.00000	.00000
14	.08869	.20947	.00326	.00129	.00000
15	.02146	.00000	.00000	.00000	.25993
16	.08106	.19638	.00153	.00057	.00000
17	.02014	.00000	.00000	.00000	.00000
18	.07475	.18547	.00072	.00026	.26091
19	.01910	.00000	.00000	.00000	.00000
20	.06944	.17620	.00034	.00012	.00000

TABLE 6.2 (continued)

(b) CONDORCET/BORDA/PLURALITY

N	(i)	(ii)	(iii)	(iv)	(v)
3	.05556	.00000	.00000	.00000	.22222
4	.08333	.18750	.07031	.04560	.00000
5	.04630	.00000	.00000	.00000	.00000
6	.05658	.07813	.01465	.00880	.12346
7	.00900	.00000	.00000	.00000	.00000
8	.06476	.03418	.00320	.00183	.00000
9	.01084	.00000	.00000	.00000	.08535
10	.03285	.01538	.00072	.00039	.00000
11	.01551	.00000	.00000	.00000	.00000
12	.02833	.00705	.00017	.00009	.06520
13	.00616	.00000	.00000	.00000	.00000
14	.03449	.00327	.00004	.00002	.00000
15	.00615	.00000	.00000	.00000	.05274
16	.02020	.00153	.00001	.00000	.00000
17	.00907	.00000	.00000	.00000	.00000
18	.01829	.00072	.00000	.00000	.04428
19	.00466	.00000	.00000	.00000	.00000
20	.02254	.00034	.00000	.00000	.00000

likelihood of Borda indecision is extremely low in non-artificial cultures. Thus, Black's procedure is virtually certain to result in a decisive outcome when N is large.

On the other hand when N is small ($\leq$ 20) the likelihood of indecision under Black's procedure is certainly not negligible. The upper limit of the likelihood of indecision varies non-monotonically with group size. When N is odd and not a multiple of 3 the upper limit of indecision under the dual amalgamation is generally less than .1. For other N $\leq$ 20 the upper limit is generally much higher. Thus, committees employing Black's procedure minimise the maximum likelihood of indecision when they consist of 5, 7, 11, 13, 17 or 19 members.

So far the dual amalgamation has been compared mainly with the Condorcet procedure. If the dual amalgamation is considered in relation to the Borda procedure the reduction in the likelihood of indecision is not nearly so large. Odd-sized groups appear to be the main beneficiaries of the arrangement. In terms of a gain in decisiveness over the primary procedure, the case for a Borda/ Condorcet amalgamation with the Borda scheme as the primary procedure is not as strong as the case for Black's procedure.

In most cultures the triple amalgamation results in a decrease in the probability of indecision obtaining under the dual amalgamation. The triple amalgamation reduces the upper limit of the likelihood of indecision for all odd N except N = 3 and N = 5. As mentioned earlier the triple amalgamation is redundant when N = 3 and N = 5. For all odd N > 3 the upper limit of the likelihood of indecision is less than .1. When N is even, the upper limit of the likelihood of indecision is the same for both the dual and triple amalgamations, being equal to

$$\frac{N!}{(N/2)!\,(N/2)!}\,(1/2)^N \quad .$$

One culture giving rise to this expression is : $q_1 = q_3 = 1/2$, $q_i = 0$, all other i. Small groups employing the triple amalgamation and wishing to avoid the worst excesses of indecision should consist of 5, 7, 11, 13, 15, 17, or 19 members.

To summarise, when N = 3 and the Condorcet method is adopted as primary procedure, both dual and triple amalgamations are redundant. When N = 4 and the Borda scheme is employed as primary procedure the dual amalgamation is redundant. When N = 5 the triple amalgamation with the plurality scheme as tertiary procedure is redundant. Although very effective when N is large, Black's procedure by no means solves the problem of indecision when N is small. The triple amalgamation further reduces the likelihood of indecision in most cultures, though when N is even the upper limit of the probability of indecision remains the same as in the dual amalgamation.

CHAPTER 7

THE PLURALITY OUTCOME AND THE WILL OF THE MAJORITY

7.1 PLURALITY - CONDORCET COINCIDENCE

The plurality outcome may be said to reflect the will of the majority when it coincides with the Condorcet outcome or, more precisely, when the same alternative emerges as both plurality and Condorcet winner. Paris (1975) termed this event plurality - Condorcet coincidence. Now, while the Condorcet procedure considers each group member's entire preference ordering of the alternatives, the plurality procedure deals only with each member's first preference. By ignoring part of the information contained in the preference orderings, the plurality method may produce a different winner from the one arrived at by the Condorcet method, when a Condorcet winner exists. This occurrence was called plurality distortion by Paris (1975). Also of interest is the situation in which a Condorcet winner exists and either the plurality winner differs from the Condorcet winner or plurality indecision obtains. This event is denoted as plurality - Condorcet disagreement.

Clearly, plurality - Condorcet disagreement subsumes plurality distortion. A closely related phenomenon, termed the Borda effect by Colman and Pountney (1978), is the occurrence of a plurality winner who receives less votes in pairwise simple majority contests than at least one other alternative. Figure 7.1 illustrates combinations of plurality and Condorcet outcomes and clarifies current terminology.

Let $PCC_m(N; q)$ denote the probability of plurality - Condorcet coincidence when N group members vote on m alternatives in culture q. Similarly, let $PCD_m(N; q)$ represent the probability of plurality - Condorcet disagreement, $PD_m(N; q)$ the probability of plurality distortion, and $BE_m(N; q)$ the probability of the Borda effect. An expression for $PD_m(N; q)$ is developed in section 7.2. The present section establishes a solution for $PCC_m(N; q)$. Expressions will not be developed for $PCD_m(N; q)$ and $BE_m(N; q)$, as it is clear from their definitions that

$$PCD_m(N; q) = 1 - CI_m(N; q) - PCC_m(N; q)$$

and

$$BE_m(N; q) = 1 - PI_m(N; q) - PCC_m(N; q) \qquad N \text{ odd.}$$

Attempts to arrive at the likelihood of plurality - Condorcet coincidence in a given situation have been undertaken by Fishburn (1974 a), Fishburn and Gehrlein (1976), and Paris (1975). Using a computer simulation procedure, Fishburn (1974 a) and Fishburn and Gehrlein (1976) investigated the operation of several voting systems, including the plurality method, in situations involving various combinations of N = 3, 5, 7, ..., 101 group members with m = 3, 4, ..., 20 alternatives. For given N and m, linear preference orderings were generated randomly for all group members, and on the basis of these preference orderings the winners according to the various voting systems were determined.

FIGURE 7.1

COMBINATIONS OF PLURALITY AND CONDORCET OUTCOMES

		PLURALITY OUTCOME: WINNER EXISTS	PLURALITY OUTCOME: INDECISION
CONDORCET OUTCOME	WINNER EXISTS	EITHER (1) SAME ALTERNATIVE IS BOTH PLURALITY AND CONDORCET WINNER / OR (2) PLURALITY AND CONDORCET WINNERS ARE NOT THE SAME ALTERNATIVE	(3) THERE IS A CONDORCET WINNER BUT NO PLURALITY WINNER
	INDECISION	(4) THERE IS A PLURALITY WINNER BUT NO CONDORCET WINNER	(5) THERE IS NEITHER A PLURALITY NOR A CONDORCET WINNER

EVENT	DESIGNATION
(1)	Plurality - Condorcet coincidence (Paris, 1975)
(2)	Plurality distortion (Paris, 1975)
(2) or (3)	Plurality - Condorcet disagreement
(2) or (4)*	Borda effect (Colman and Pountney, 1978)

*Event (4) implies the Borda effect if N is odd. However, if N is even Condorcet indecision may take the form of a tie which precludes the Borda effect.

Note that randomness implies an equiprobable culture. This procedure was repeated 1000 times. An efficiency index was computed for each system expressing, roughly speaking, the percentage of occasions on which the system produced the same winner as the Condorcet method given that a Condorcet winner emerged. In an equiprobable culture for given N, it was established that plurality efficiency decreases as m increases. In the same culture for given m, there is also a reduction in plurality efficiency as N increases. No other identifiable cultures were considered by the authors, though efficiency indices each of which was based on 1000 unidentifiable cultures, were reported.

Paris (1975) employed a computer search technique, of the kind used by Garman and Kamien (1968) and Niemi and Weisberg (1968) in their analyses of the cyclical majority problem, to elicit those distributions of the voters over the preference orderings which give rise to plurality - Condorcet coincidence. Applying the results to the multinomial social choice model, he succeeded in obtaining the likelihood of plurality - Condorcet coincidence in case $m = 3$ for a number of cultures. However, the enormous amount of computer time demanded by this approach prevented consideration of group sizes larger than $N = 17$. For reasons that are less clear group sizes of $N < 11$ were also ignored. As in the Fishburn simulation studies only odd-sized groups were examined.

The strength of the computer search procedure is that it provides accurate probability values. Unfortunately, in this respect Paris's results are somewhat lacking. Elementary checks indicate that a programming error has occurred. A major weakness of the approach is that computer time limitations restrict attention to small group sizes. By contrast, the computer simulation approach, although it cannot guarantee accuracy, can provide a global impression of the efficiency of the plurality method for a variety of values of N and m. Time limitations may be overcome by accepting a reduction in accuracy, or accuracy may be improved by lengthening the duration of the simulation.

The path followed in the present analysis leads to the development of an explicit, closed- form expression for the probability or plurality - Condorcet coincidence in case m = 3. Precise probability values are obtained for group sizes up to N = 1001 in a variety of cultures. Both odd- and even-sized groups are examined. However, the principal advantage of the present solution over previous ones is that, by means of the obtained formula, implications of the phenomenon under scrutiny may be drawn out and insight gained into the conditions influencing the range of its possible behaviour.

Now, $PCC_m(N; q)$ represents the probability of plurality-Condorcet coincidence when N group members are voting on m alternatives in culture q. If $PCC_m(A_i; N; q)$ denotes the probability that A_i is both the plurality and the Condorcet winner then we have

$$PCC_m(N; q) = \sum_{i=1}^{m} PCC_m(A_i; N; q) \tag{7.1}$$

An expression for $PCC_m(A_i; N; q)$ is derived subsequently in case m = 3.

Let g_L represent a lower limit, and g_U an upper limit, of exactly g first preference votes. It is useful to employ $PCC_m(A_i; N; q; (k+1)_L)$ to denote the probability that A_i is both the plurality and the Condorcet winner with at least k + 1 first preference votes, where 2k = N if N is even and 2k + 1 = N if N is odd. Likewise, define $PCC_m(A_i; N; q; k_U)$ as the probability that A_i is both the plurality and the Condorcet winner with k or fewer first preference votes. Thus,

$$PCC_m(A_i;\ N;\ q) = PCC_m(A_i;\ N;\ q;\ (k+1)_L) + PCC_m(A_i;\ N;\ q;\ k_U) \tag{7.2}$$

Similarly, the probability that A_i is the plurality winner may be decomposed into

$$P_m(A_i;\ N;\ p) = P_m(A_i;\ N;\ p;\ (k+1)_L) + P_m(A_i;\ N;\ p;\ k_U) \tag{7.3}$$

where the two expressions on the right hand side represent the probability that A_i is the plurality winner with, respectively, a lower limit of k + 1 votes and an upper limit of k votes.

Define

$$PCC_m(N;\ q;\ (k+1)_L) = \sum_{i=1}^{m} PCC_m(A_i;\ N;\ q;\ (k+1)_L)$$

$$PCC_m(N;\ q;\ k_U) = \sum_{i=1}^{m} PCC_m(A_i;\ N;\ q;\ k_U)$$

and

$$P_m(N;\ p;\ (k+1)_L) = \sum_{i=1}^{m} P_m(A_i;\ N;\ p;\ (k+1)_L)$$

$$P_m(N;\ p;\ k_U) = \sum_{i=1}^{m} P_m(A_i;\ N;\ p;\ k_U) \quad .$$

Therefore, we have

$$PCC_m(N;\ q) = PCC_m(N;\ q;\ (k+1)_L) + PCC_m(N;\ q;\ k_U) \tag{7.4}$$

and

$$P_m(N;\ p) = P_m(N;\ p;\ (k+1)_L) + P_m(N;\ p;\ k_U) \tag{7.5}$$

When an alternative becomes plurality winner with at least k + 1 first preference votes, where 2k = N if N is even and 2k + 1 = N if N is odd, that alternative is necessarily also the Condorcet winner. That is,

$$PCC_m(A_i;\ N;\ q;\ (k+1)_L) = P_m(A_i;\ N;\ p;\ (k+1)_L) \tag{7.6}$$

and

$$PCC_m(N;\ q;\ (k+1)_L) = P_m(N;\ p;\ (k+1)_L) \tag{7.7}$$

Consider alternative A_1 in case m = 3. The probability that A_1 is the plurality winner with more than k first preference votes is obtained by modifying expression (3.1) to take into account the new lower limit for a_1. Thus,

$$P_3(A_1;\ N;\ p;\ (k+1)_L) = \sum_{a_1=k+1}^{N} \sum_{a_2=0}^{N-a_1} \frac{N!}{a_1!a_2!a_3!} p_1^{a_1} p_2^{a_2} p_3^{a_3}$$

$$= \sum_{a_1=k+1}^{N} \frac{N!}{a_1!(N-a_1)!} p_1^{a_1} (p_2 + p_3)^{N-a_1} \tag{7.8}$$

by the binomial theorem. Similar expressions may be obtained for $P_3(A_2;\ N;\ p;\ (k+1)_L)$ and $P_3(A_3;\ N;\ p;\ (k+1)_L)$. Hence, by (7.6) and (7.8)

$$PCC_3(A_1;\ N;\ q;\ (k+1)_L)$$
$$= \sum_{a_1=k+1}^{N} \frac{N!}{a_1!(N-a_1)!} p_1^{a_1} (p_2 + p_3)^{N-a_1} \tag{7.9}$$

Also, by definition

$$PCC_3(N;\ q;\ (k+1)_L)$$
$$= \sum_{i=1}^{3} PCC_3(A_i;\ N;\ q;\ (k+1)_L)$$
$$= \sum_{a_1=k+1}^{N} \frac{N!}{a_1!(N-a_1)!} \Big[p_1^{a_1} (p_2 + p_3)^{N-a_1} + p_2^{a_1} (p_1 + p_3)^{N-a_1} + p_3^{a_1} (p_1 + p_2)^{N-a_1} \Big] \tag{7.10}$$

where a_1, the number of votes received by A_1, has been substituted for a_2 and a_3 because the number of votes received by a plurality winner has the same range of values irrespective of the identity of the alternative.

Now, by a complementary adaptation of expression (3.1) we obtain the probability that A_1 is the plurality winner with k or fewer first preference votes,

$$p_3(A_1;\ N;\ p;\ k_U) = \overset{2}{\Sigma} \frac{N!}{a_1!a_2!a_3!} p_1^{a_1} p_2^{a_2} p_3^{a_3} \qquad (7.11)$$

where $\overset{2}{\Sigma}$ is a 2-fold summation whose variables of summation have the following limits

$$\left\{(N+4)/3\right\} \leq a_1 \leq k \qquad (7.12)$$

$$N - 2a_1+1 \leq a_2 \leq a_1 - 1$$

Recalling that $p_2 = q_3 + q_4$ and $p_3 = q_5 + q_6$, and likewise that $a_2 = x_3 + x_4$ and $a_3 = x_5 + x_6$, expression (7.11) may be expanded by applying the binomial theorem twice so that

$$p_3(A_1;\ N;\ p;\ k_U)$$

$$= \overset{4}{\Sigma} \frac{N!}{a_1!x_3!x_4!x_5!x_6!} p_1^{a_1} q_3^{x_3} q_4^{x_4} q_5^{x_5} q_6^{x_6} \qquad (7.13)$$

where $\overset{4}{\Sigma}$ is a 4-fold summation whose variables of summation have the following limits

$$\left\{(N+4)/3\right\} \leq a_1 \leq k \qquad (7.14)$$

$$N - 2a_1+1 \leq a_2 \leq a_1 - 1$$

$$0 \leq x_3 \leq a_2$$

$$0 \leq x_5 \leq N - a_1 - a_2$$

and where $x_4 = a_2 - x_3$ and $x_6 = N - a_1 - a_2 - x_5$.

Now, for A_1 to be Condorcet winner, A_1 must defeat A_2 and A_3 in pairwise simple majority contests. That is, the following inequalities must hold

$$\begin{aligned} a_1 + x_5 &\geq k + 1 \\ a_1 + x_3 &\geq k + 1 \end{aligned} \tag{7.15}$$

In other words, if A_1 is to be Condorcet winner both x_3 and x_5 must have a lower limit equal to $k + 1 - a_1$. Modifying the limits of summation (7.14) of formula (7.13) accordingly, we obtain an expression for the probability that A_1 is both the plurality and the Condorcet winner with k or fewer first preference votes. Thus,

$$PCC_3(A_1;\ N;\ q;\ k_U)$$

$$= \overset{4}{\Sigma} \frac{N!}{a_1!x_3!x_4!x_5!x_6!} p_1^{a_1} q_3^{x_3} q_4^{x_4} q_5^{x_5} q_6^{x_6} \tag{7.16}$$

where $\overset{4}{\Sigma}$ is a 4-fold summation whose variables of summation have the following limits

$$\begin{aligned} \{(N+4)/3\} &\leq a_1 \leq k \\ N - 2a_1+1 &\leq a_2 \leq a_1 - 1 \\ k+1 - a_1 &\leq x_3 \leq a_2 \\ k+1 - a_1 &\leq x_5 \leq N - a_1 - a_2 \end{aligned} \tag{7.17}$$

and where $x_4 = a_2 - x_3$ and $x_6 = N - a_1 - a_2 - x_5$. Similar solutions may be found for $PCC_3(A_2;\ N;\ q;\ k_U)$ and $PCC_3(A_3;\ N;\ q;\ k_U)$.

The substitution of expressions (7.9) and (7.16) in (7.2) gives the probability that A_1 is both the plurality and Condorcet winner, i.e.

$$PCC_3(A_1;\ N;\ q) = \sum_{a_1=k+1}^{N} \frac{N!}{a_1!(N-a_1)!} p_1^{a_1} (p_2 + p_3)^{N-a_1}$$

$$+ \sum^{4} \frac{N!}{a_1!x_3!x_4!x_5!x_6!} p_1^{a_1} q_3^{x_3} q_4^{x_4} q_5^{x_5} q_6^{x_6} \qquad (7.18)$$

where $\sum^{4}$ is a 4-fold summation whose variables of summation have limits given by (7.17).

Finally, by (7.4), (7.10) and (7.18) we arrive at the probability of plurality - Condorcet coincidence

$$PCC_3(N;\ q)$$

$$= \sum_{a_1=k+1}^{N} \frac{N!}{a_1!(N-a_1)!}\Big[p_1^{a_1} (p_2 + p_3)^{N-a_1} + p_2^{a_1} (p_1 + p_3)^{N-a_1} + p_3^{a_1} (p_1 + p_2)^{N-a_1} \Big]$$

$$+ \sum^{4} \frac{N!}{a_1!x_3!x_4!x_5!x_6!}\Big[p_1^{a_1} q_3^{x_3} q_4^{x_4} q_5^{x_5} q_6^{x_6} + q_1^{x_5} q_2^{x_6} p_2^{a_1} q_5^{x_4} q_6^{x_3} + q_1^{x_4} q_2^{x_3} q_3^{x_6} q_4^{x_5} p_3^{a_1} \Big] \qquad (7.19)$$

where $\sum^{4}$ is a 4-fold summation whose variables of summation have limits given by (7.17). It will be noted that expressions for $PCC_3(A_i;\ N;\ q;\ k_U)$, i = 1, 2, 3, have been brought together under a single 4-fold summation sign. This is achieved by interchanging the labels for the alternatives thereby establishing for each preference ordering when A_1 is plurality-Condorcet winner, the corresponding preference ordering when a different alternative is plurality - Condorcet winner. Since the number of votes received by corresponding preference orderings, under their respective winners, must assume the same range of values, we may substitute a_1, x_3, x_4, x_5 and x_6 for their respective counterparts when A_2 and A_3 are plurality - Condorcet winners. Hence, only a single 4-fold summation is required.

7.2 PLURALITY DISTORTION

The probability of plurality distortion, that is, the likelihood that the plurality method selects a non-Condorcet winner when a Condorcet winner exists, has received preliminary investigation by Paris (1975). Using a computer search routine he obtained probability values in case m = 3 for odd - sized groups of between N = 11 and N = 29 members in a number of cultures. As in the case of plurality - Condorcet coincidence, some of these values prove to be inaccurate.

The solution developed in the present section permits the calculation of the likelihood of plurality distortion in case m = 3 for both odd - and even - sized groups of up to N = 1001 members in any culture.

Now, $PD_m(N; q)$ represents the probability of plurality distortion when N group members vote on m alternatives in culture q. If $PD_m(A_i, A_j; N; q)$ denotes the plurality that A_i is the plurality winner while A_j is the Condorcet winner then we have

$$PD_m(N; q) = \sum_{\substack{i,j=1 \\ i \neq j}}^{m} PD_m(A_i, A_j; N; q) \tag{7.20}$$

An expression for $PD_m(A_i, A_j; N; q)$ is developed in the case of three alternatives.

For plurality distortion to occur the plurality winner must receive N - k - 1 or fewer first preference votes, where 2k = N is even and 2k + 1 = N if N is odd. Consider alternative A_1 in case m = 3. The probability that A_1 is the plurality winner with N - k - 1 or fewer first preference votes may be obtained from equation (7.11) by replacing a_1's upper limit of **k** in (7.12) with N - k - 1, giving

$$p_3(A_1;\ N;\ p;\ (N-k-1)_U) = \sum^{2} \frac{N!}{a_1!a_2!a_3!}\, p_1^{a_1}\, p_2^{a_2}\, p_3^{a_3} \tag{7.21}$$

where $\sum^{2}$ is a 2-fold summation whose variables of summation have the following limits

$$\begin{aligned} \{(N+4)/3\} &\leq a_1 \leq N-k-1 \\ N-2a_1+1 &\leq a_2 \leq a_1 - 1 \quad . \end{aligned} \tag{7.22}$$

Recalling that $p_1 = q_1 + q_2$ and $p_2 = q_3 + q_4$, and likewise that $a_1 = x_1 + x_2$ and $a_2 = x_3 + x_4$, expression (7.21) may be expanded by applying the binomial theorem twice, so that

$$p_3(A_1;\ N;\ p;\ (N-k-1)_U) = \sum^{4} \frac{N!}{x_1!x_2!x_3!x_4!a_3!}\, q_1^{x_1}\, q_2^{x_2}\, q_3^{x_3}\, q_4^{x_4}\, p_3^{a_3} \tag{7.23}$$

where $\sum^{4}$ is a 4-fold summation whose variables of summation have the following limits

$$\begin{aligned} \{(N+4)/3\} &\leq a_1 \leq N-k-1 \\ N-2a_1+1 &\leq a_2 \leq a_1 - 1 \\ 0 &\leq x_1 \leq a_1 \\ 0 &\leq x_3 \leq a_2 \quad . \end{aligned} \tag{7.24}$$

Let us suppose that alternative A_3 is the Condorcet winner while A_1 is the plurality winner. For A_3 to be the Condorcet winner A_3 must defeat A_1 and A_2 in pairwise simple majority contests. That is, the following inequalities must hold :

$$\begin{aligned} a_1 + x_3 &\leq N-k-1 \\ a_2 + x_1 &\leq N-k-1 \end{aligned} \tag{7.25}$$

Therefore, if A_1 is to be the plurality winner while A_3 is the Condorcet winner then x_1 must have an upper limit of $N-k-1-a_2$, and x_3 must have an upper limit of $N - k - 1 - a_1$. Modifying the limits of summation (7.24) of formula (7.23) accordingly, we obtain an expression for the probability that A_1 is the plurality winner while A_3 is the Condorcet winner. Thus,

$$PD_3(A_1, A_3; N; q)$$

$$= \sum^4 \frac{N!}{x_1!x_2!x_3!x_4!a_3!} q_1^{x_1} q_2^{x_2} q_3^{x_3} q_4^{x_4} p_3^{a_3} \qquad (7.26)$$

where $\sum^4$ is a 4-fold summation whose variables of summation have the following limits

$$\begin{aligned} \{(N+4)/3\} &\leq a_1 \leq N - k - 1 \\ N - 2a_1+1 &\leq a_2 \leq a_1 - 1 \\ 0 &\leq x_1 \leq N - k - 1 - a_2 \\ 0 &\leq x_3 \leq N - k - 1 - a_1 \end{aligned} \qquad (7.27)$$

Similar expressions may be obtained for $PD_3(A_1, A_2; N; q)$, $PD_3(A_2, A_1; N; q)$, $PD_3(A_2, A_3; N; q)$, $PD_3(A_3, A_1; N; q)$ and $PD_3(A_3, A_2; N; q)$.

Finally, by (7.20) we arrive at an expression for the probability of plurality distortion:

$$\begin{aligned} PD_3(N; q) = \sum^4 \frac{N!}{x_1!x_2!x_3!x_4!a_3!} \Big[& q_1^{x_1} q_2^{x_2} q_3^{x_3} q_4^{x_4} p_3^{a_3} \\ & + q_1^{x_2} q_2^{x_1} p_2^{a_3} q_5^{x_3} q_6^{x_4} + p_1^{a_3} q_3^{x_2} q_4^{x_1} q_5^{x_4} q_6^{x_3} \\ & + q_1^{x_3} q_2^{x_4} q_3^{x_1} q_4^{x_2} p_3^{a_3} + p_1^{a_3} q_3^{x_4} q_4^{x_3} q_5^{x_2} q_6^{x_1} \\ & + q_1^{x_4} q_2^{x_3} p_2^{a_3} q_5^{x_1} q_6^{x_2} \Big] \end{aligned} \qquad (7.28)$$

where $\sum^4$ is a 4-fold summation whose variables of summation have limits given by (7.27). It will be observed that expressions for $PD_3(A_i, A_j; N; q)$, $i,j = 1, 2, 3$, $i \neq j$, have been grouped together under a single 4-fold summation sign. This is achieved by interchanging the labels for the alternatives thereby determining for each preference ordering, when A_1 is the plurality winner and A_3 is the Condorcet winner, the corresponding preference ordering under different plurality or Condorcet winners. Since the number of votes received by corresponding preference orderings, under their respective winners, must assume the same range of values, we may substitute x_1, x_2, x_3, x_4 and a_3 for their respective counterparts under each combination of plurality and Condorcet winners. In this way only a single 4-fold summation is required.

7.3 EFFECT OF GROUP SIZE, NUMBER OF ALTERNATIVES, AND CULTURE ON THE LIKELIHOOD OF AGREEMENT BETWEEN PLURALITY AND CONDORCET OUTCOMES

Paris (1975) points to the importance for democratic theory of several, as yet unexplored, avenues of research:

(i) the calculation of the probability of plurality distortion for large group sizes;

(ii) the calculation of the probability of plurality distortion when more than three alternatives are being voted on;

(iii) the formulation of general principles which will permit the determination of those cultures in which plurality distortion is likely to occur.

The present section aims to throw light on all three issues: firstly, by examining the asymptotic behaviour of the probability of plurality distortion as N becomes large; secondly, by using the formulae derived in sections 7.1 and 7.2 to provide exact probability values in case m = 3 for groups of up to N = 1001 members in a number of cultures; and, thirdly, by employing a computer search procedure to obtain exact probability values in case m = 4 for groups of up to N = 20 members in a number of cultures.

When each of a number of events has a probability of occurrence equal to unity, (zero), then the probability of their joint occurrence must also equal unity (zero). It was established in chapter 3 that as N becomes large the probability that A_i is the plurality winner behaves in the following manner:

$$\begin{aligned} P_m(A_i; N; p) &= 1 \qquad p_i > p_j, \quad \text{all } j \neq i \,; \\ &= 0 \qquad p_i < p_j, \quad \text{some } j \neq i \,. \end{aligned} \tag{7.29}$$

In chapter 4 we found that as N becomes large the probability that A_i is the Condorcet winner is given by

$$\begin{aligned} C_m(A_i; N; q) &= 1 \qquad q_{ij} > 1/2, \text{ all } j \neq i \,; \\ &= 0 \qquad q_{ij} < 1/2, \text{ some } j \neq i \,. \end{aligned} \tag{7.30}$$

It follows that as N becomes large the probability that A_i is both the plurality and the Condorcet winner is

$$\begin{aligned} PCC_m(A_i; N; q) &= 1 \qquad p_i > p_j \text{ and } q_{ij} > 1/2, \quad \text{all } j \neq i \,; \\ &= 0 \qquad p_i < p_j \text{ or } q_{ij} < 1/2, \quad \text{some } j \neq i \,. \end{aligned} \tag{7.31}$$

Likewise, as N becomes large the probability that A_i is the plurality winner while A_h is the Condorcet winner equals

$$PD_m(A_i, A_h; N; q) = 1 \qquad p_i > p_j, \text{ all } j \neq i, \text{ and } q_{hj} > 1/2, \text{ all } j \neq h ;$$
$$= 0 \qquad p_i < p_j, \text{ some } j \neq i, \text{ or } q_{hj} < 1/2, \text{ some } j \neq h . \qquad (7.32)$$

Now, if we ignore unrealistic, or "pathological" (May, 1971), cultures in which the equalities, $q_{ij} = 1/2$ or $p_i = p_j$, some $j \neq i$, hold exactly then only one of three mutually exclusive and exhaustive outcomes can occur in a given culture as N becomes large. Either plurality - Condorcet coincidence, plurality distortion, or Condorcet indecision occurs in a given culture with probability equal to unity in the asymptote. In other words, using the results (7.30), (7.31), and (7.32), all cultures may be divided into three exhaustive and non-overlapping types, namely, those in which $PCC_m(N; q)$, $PD_m(N; q)$, and $CI_m(N; q)$, respectively, attain unity as N becomes large. Table 7.1 delineates the three types of culture in case m = 3. A culture is specified in terms of pairwise probabilities, q_{12}, q_{13}, q_{23}, and first preference probabilities, p_1, p_2, p_3 . Type C cultures result in plurality - Condorcet coincidence, type D cultures produce plurality distortion, and type I cultures give rise to Condorcet indecision. In Table 7.1, type D cultures are further subdivided into D_1 and D_2. In type D_1 only the Condorcet winner is preferred by a simple majority to the plurality winner. In type D_2 both the Condorcet winner and the other alternative are preferred in a simple majority sense to the plurality winner. That is, in type D_2 cultures the plurality winner is actually certain to be the least preferred alternative as N becomes large.

TABLE 7.1

CLASSIFICATION OF CULTURES IN CASE m = 3 ACCORDING TO WHETHER PLURALITY - CONDORCET COINCIDENCE (C), PLURALITY DISTORTION (D), OR CONDORCET INDECISION (I) OCCURS WITH PROBABILITY EQUAL TO UNITY WHEN N IS LARGE.

		PAIRWISE PROBABILITIES							
	$q_{12} - .5$:	+	+	+	+	-	-	-	-
	$q_{13} - .5$:	+	+	-	-	+	+	-	-
	$q_{23} - .5$:	+	-	+	-	+	-	+	-
LARGEST FIRST PREFERENCE PROBABILITY ($p_i < 1/2$, $i = 1, 2, 3$)	p_1	C	C	I	D_1	D_1	I	D_2	D_2
	p_2	D_1	D_2	I	D_2	C	I	C	D_1
	p_3	D_2	D_1	I	C	D_2	I	D_1	C

NOTE (i) The subscript in D_1, D_2 refers to the number of alternatives who are actually preferred in a simple majority sense to the plurality winner.

(ii) When $p_i > 1/2$, plurality - Condorcet coincidence is assured for alternative A_i in the asymptote.

Table 7.1 may be used to determine in case $m = 3$ whether a given culture will result in conflict or agreement between the plurality and Condorcet outcomes when N is large. For example, consider culture $q = (.05, .20, .15, .25, .30, .05)$. The pairwise probabilities are

$$q_{12} = .05 + .20 + .30 = .55$$
$$q_{13} = .05 + .20 + .15 = .40$$
$$q_{23} = .05 + .15 + .25 = .45$$

Therefore,

$$q_{12} - .5 = + .05$$
$$q_{13} - .5 = - .1$$
$$q_{23} - .5 = - .05$$

This pattern of + - - in the pairwise probabilities together with the fact that the largest first preference probability is $p_2 = .15 + .25 = .4$ directs our attention to column 4, row 2 of Table 7.1. The entry there informs us that when N is large plurality distortion is inevitable in this culture. Alternative A_2 is certain to be plurality winner while A_3 is equally certain to be Condorcet winner.

Similar, if somewhat larger, tables may readily be constructed on the same principles when $m > 3$.

Paris (1975) offers the "tentative" hypothesis that "the probability of plurality distortion is inversely related to the probability of the paradox". Since the paradox of voting does not occur in single - peaked cultures (Black, 1958 ; Sen 1970), the probability of plurality distortion in such cultures should always be fairly high if Paris's hypothesis holds. Consider the single - peaked culture $q = (q_1, 0, q_3, q_4, 0, q_6)$. If $p_i > 1/2$ then, as in Table 7.1, plurality - Condorcet coincidence is assured for alternative A_i in the asymptote. However, if

$p_i < 1/2$, $i = 1, 2, 3$, then $q_{12} < 1/2$ and $q_{23} > 1/2$. Therefore, only A_2 can become Condorcet winner in the asymptote. Accordingly, Table 7.1 reduces from eight columns to two columns, namely, column 5 and column 7. From column 5 and column 7 it is clear that plurality distortion can asymptote either to unity or to zero in a single - peaked culture. Thus, while it is true that as the probability of the paradox increases so the probability of plurality distortion decreases (simply because the likelihood of any Condorcet winner emerging decreases), a reduction in the probability of the paradox does not entail an increase in the probability of plurality distortion. Paris's hypothesis of an inverse relationship between the two probabilities must therefore be rejected.

Paris also raises a methodological point with which I should like to take issue. He reasons that, because it is relatively unusual for a candidate in American state elections to receive more than half the first preference votes, "the comparison of plurality distortion and plurality - Condorcet coincidence should be confined to voter distributions in which no candidate receives more than half the vote".

Now, if no candidate receives more than half the vote in a state election, this strongly suggests that $p_i < 1/2$, $i = 1, 2, \ldots, m$. The reason for this is that the probability that A_i receives $N.v < a_i < N.w$ first preference votes, where $0 \leq v < w \leq 1$, is given by the binomial distribution

$$p(Nv < a_i < Nw) = \sum_{a_i=Nv}^{Nw} \frac{N!}{a_i!(N-a_i)!} p_i^{a_i} (1-p_i)^{N-a_i} \qquad (7.33)$$

From the result contained in expression (3.6) it follows that as N tends to infinity

$$p(Nv < a_i < Nw) = 1 \quad \text{if } v < p_i < w ; \qquad (7.34)$$
$$= 0 \quad \text{otherwise.}$$

From (7.34) it is evident that if $p_i < 1/2$, $i = 1, 2, \ldots, m$, then the probability that a plurality winner receives less than half the first preference votes tends to unity as N becomes large (e.g. the size of a state electorate). Likewise, if $p_i > 1/2$, the probability that A_i is the plurality winner with more than half the first preference votes tends to unity as N becomes as large as a state electorate. From the above considerations it would appear that in state elections $p_i < 1/2$, $i = 1, 2, \ldots, m$. Therefore, the likelihood of plurality - Condorcet coincidence with the plurality winner receiving more than half the first preference votes equals zero. Hence, with large N no modification of the kind suggested by Paris is necessary.

When N is small the situation is somewhat different. Paris terms the case in which an alternative receives more than half the first preference votes an "unusual distribution". However, it is only unusual for large groups with $p_i < 1/2$, $i = 1, 2, \ldots, m$. It is contradictory to describe the event as unusual for small groups when the probability associated with its occurrence is sizeable! Therefore, for small N also Paris's recommendation is contraindicated.

In an even more recent investigation of the correspondence between plurality and Condorcet outcomes Colman and Pountney (1978) define the Borda effect as the circumstance in which either one (weak Borda effect) or both (strong Borda effect) of the plurality losers is preferred by a simple majority to the plurality winner. Since the Borda effect can occur simultaneously with the paradox of voting, the concept is not synonymous with plurality distortion, although clearly the two overlap. As an example of its asymptotic behaviour, if p_1 is the largest plurality probability the Borda effect occurs with probability equal to unity with sufficiently large N if either $q_{12} < 1/2$ or $q_{13} < 1/2$.

In an analysis of the results of the 1966 British General Election, Colman and Pountney found evidence of the occurrence of the Borda effect. They estimated that in 15 out of 261 three-cornered contests a majority of voters actually preferred a particular plurality loser to the plurality winner. However, this figure may be an overestimate. In their analysis Colman and Pountney employed estimates of the second preference probabilities of the voters, derived from the national survey conducted by Butler and Stokes (1971). The same national estimates were used to analyse the results in every constituency because estimates for individual constituencies were not available. Although the Labour party went on to win the election almost all instances of the Borda effect occurred in constituencies where the winner belonged to the Conservative or Liberal parties. However, it is probably in these constituencies, where the national trend is reversed, that the estimates of second preference probabilities are least accurate. The frequency of occurrence of the Borda effect may, therefore, have been overestimated. Clearly, there is an urgent need for information at the constituency level on voters' preference orderings in order that the extent of the Borda effect and of plurality distortion in elections may be accurately gauged.

Turning our attention to more moderate values of N, Table 7.2 provides the likelihood of plurality - Condorcet coincidence and of plurality distortion in case m = 3 for groups of up to 1001 members in each of three cultures. The tabled values were calculated by means of formulae (7.19) and (7.28). The cultures considered are : (i) the equiprobable culture, (ii) the culture in which the likelihood of the paradox of voting reaches its maximum, and (iii) a single - peaked culture.

The noticeable difference between probability values for odd - and even - sized groups in Table 7.2 arises because of the higher likelihood of Condorcet indecision in even - sized groups. When a Condorcet winner fails to emerge neither plurality - Condorcet coincidence nor plurality distortion can occur. Also, because $CI_3(N; q)$ and $PI_3(N; p)$ are often sizeable when N is small, $PCC_3(N; q)$ and $PD_3(N; q)$ generally do not sum to unity except when N is large.

In groups of N = 3, 4, 5, 6 or 8 members plurality distortion cannot occur because an alternative cannot become plurality winner with less than half the first preference votes.

In culture (i) the likelihood of plurality - Condorcet coincidence remains at around .7 for all odd values of N. When N is even the likelihood climbs as group size increases from around .5 to a limiting value of approximately .7. In the same culture, the likelihood of plurality distortion increases with N from an initial value of zero to a limiting value of around .2.

In culture (ii) the set of preference ordering probabilities has the form $q = (q_1, 0, 0, q_4, q_5, 0)$. That is, we have a reduced set of preference orderings :

$$\begin{array}{ccc} A_1 & A_2 & A_3 \\ A_2 & A_3 & A_1 \\ a_3 & A_1 & A_2 \end{array} .$$

If the plurality winner receives k or fewer first preference votes, where 2k = N if N is even and 2k + 1 = N if N is odd, then Condorcet indecision must occur. Otherwise, plurality - Condorcet coincidence prevails. Therefore, in culture

TABLE 7.2

PROBABILITY OF PLURALITY DISTORTION (PD) AND OF PLURALITY-CONDORCET COINCIDENCE (PCC) FOR N GROUP MEMBERS AND m = 3 ALTERNATIVES IN EACH OF THREE CULTURES :

(i) q = (1/6, 1/6, 1/6, 1/6, 1/6, 1/6),

(ii) q = (1/3, 0, 0, 1/3, 1/3, 0), and

(iii) q = (1/3, 0, 1/6, 1/6, 0, 1/3).

	Culture (i)		Culture (ii)		Culture (iii)	
N	PD	PCC	PD	PCC	PD	PCC
3	.00000	.77778	.00000	.77778	.00000	.77778
4	.00000	.44444	.00000	.33333	.00000	.48148
5	.00000	.62963	.00000	.62963	.00000	.62963
6	.00000	.48560	.00000	.30041	.00000	.46502
7	.07202	.68193	.00000	.51989	.19204	.61591
8	.00000	.48388	.00000	.26383	.00000	.41320
9	.08102	.68660	.00000	.43454	.25606	.56257
10	.01667	.51109	.00000	.22969	.14225	.42885
11	.06999	.65110	.00000	.36626	.26080	.49666
12	.02323	.52714	.00000	.19934	.20864	.41146
13	.10703	.67469	.00000	.31062	.35411	.48767
14	.02369	.52500	.00000	.17285	.22854	.37753
15	.11033	.67812	.00000	.26469	.38927	.45933
16	.03891	.54146	.00000	.14989	.31700	.38433
17	.09937	.65900	.00000	.22643	.38763	.42024
18	.04363	.55058	.00000	.13004	.35782	.37296
19	.12434	.67401	.00000	.19430	.44541	.41700
20	.04256	.54814	.00000	.11291	.36561	.34987
50	.09217	.59605	.00000	.01475	.57249	.30691
51	.15832	.67740	.00000	.02066	.59185	.31659
100	.12724	.62396	.00000	.00060	.62073	.31118
101	.17199	.67751	.00000	.00082	.61577	.30870
500	.17475	.65927	.00000	.00000	.64446	.32223
501	.20027	.68518	.00000	.00000	.64558	.32279
1000	.18827	.66855	.00000	.00000	.65159	.32579
1001	.20710	.68535	.00000	.00000	.65104	.32552

$q = (q_1, 0, 0, q_4, q_5, 0)$ we find

$$PD_3(N; q) = 0 \tag{7.35}$$

and

$$PCC_3(N; q) = P_3(N; p; (k+1)_L) \tag{7.36}$$

Since $p_1 = q_1$, $p_2 = q_4$, and $p_3 = q_5$, it follows that if any $p_i > 1/2$ then as N becomes large $PCC_3(N; q)$ tends to unity ; otherwise $PCC_3(N; q)$ diminishes to zero. Culture (ii) is of the latter type.

Culture (iii) in Table 7.2 is single - peaked and has the form $q = (q_1, 0, q_3, q_4, 0, q_6)$. Clearly, A_2 is the central alternative on the underlying dimension. The reduced set of preference orderings is

A_1	A_2	A_2	A_3
A_2	A_1	A_3	A_2
A_3	A_3	A_1	A_1 .

Now, because the paradox of voting cannot occur in a single - peaked culture we have the following relationship when N is odd :

$$P_m(N; p) = PCC_m(N; q) + PD_3(N; q) \quad \text{all odd } N \tag{7.37}$$

Also, when $N = 2k + 1$ is odd, consideration of the reduced set of preference orderings reveals that if either A_1 or A_3 is the plurality winner with k or fewer first preference votes then plurality distortion is bound to occur. Otherwise, plurality - Condorcet coincidence obtains. Therefore, in culture $q = (q_1, 0, q_3, q_4, 0, q_6)$ when N is odd

$$PD_3(N; q) = P_3(A_1; N; p; k_U) + P_3(A_3; N; p; k_U) \tag{7.38}$$

and

$$PCC_3(N; q) = P_3(N; p; (k+1)_L) + P_3(A_2; N; p; k_U) \tag{7.39}$$

Evidently, in a single - peaked culture when $m = 3$, the likelihood of plurality distortion and plurality - Condorcet coincidence depend entirely on first preference probabilities. Thus, in culture (iii), because $p_i = 1/3$, $i = 1, 2, 3$, it follows from (7.34) that as N becomes large $P_3(N; p; (k+1)_L)$ tends to zero. Also, because of the symmetry in the p_i, $P_3(A_i; N; p; k_U)$ approaches a limiting value of 1/3 as N becomes large. Therefore, in culture (iii), in the asymptote $PD_3(\infty; q) = 2/3$ and $PCC_3(\infty; q) = 1/3$.

Now, exact equiprobability among the p_i, $i = 1, 2, 3$, is a highly unlikely state of affairs. Table 7.3 illustrates the behaviour of $PD_3(N; q)$ as a function of N in two single - peaked cultures of the same form as culture (iii) but with first preference probabilities marginally different from equiprobability. Both cultures share the same set of first preference probabilities, although these probabilities are associated with different alternatives. In culture (i), Table 7.3, the central alternative A_2 has the lowest probability, whereas in culture (ii) one of the extreme alternatives, A_3, has the lowest probability.

When N is small ($N \leq 20$) the probability of plurality distortion is of much the same magnitude in the three cultures : culture (iii), Table 7.2; culture (i), Table 7.3; and culture (ii), Table 7.3. As N becomes large, the probability of plurality distortion in cultures (i) and (ii) of Table 7.3 tends, respectively, to unity and to zero. Of course, the asymptotic value of $PD_3(N; q)$ in a given culture may be ascertained from Table 7.1. However, to determine the manner in which $PD_3(N; q)$ approaches the asymptote in a single - peaked culture, equation (7.38) is examined.

TABLE 7.3

PROBABILITY OF PLURALITY DISTORTION FOR N GROUP MEMBERS AND m = 3 ALTERNATIVES IN TWO SINGLE - PEAKED CULTURES WITH FIRST PREFERENCE PROBABILITIES WHICH DEPART MARGINALLY FROM EQUIPROBABILITY.

THE CULTURES, BOTH OF WHICH HAVE A_2 AS THE CENTRAL ALTERNATIVE, ARE :

(i) p = (10/30, 9/30, 11/30), AND

(ii) p = (10/30, 11/30, 9/30).

N	Culture (i)	Culture (ii)
3	.00000	.00000
4	.00000	.00000
5	.00000	.00000
6	.00000	.00000
7	.19763	.17881
8	.00000	.00000
9	.26972	.23220
10	.14493	.13113
11	.28167	.23083
12	.21758	.18731
13	.38124	.31171
14	.24440	.20029
15	.42451	.33695
16	.33875	.27603
17	.43092	.32915
18	.38761	.30645
19	.49461	.37585
20	.40386	.30733
50	.70389	.41588
51	.73053	.42863
100	.82419	.38662
101	.82264	.38205
500	.96598	.19088
501	.96630	.19081
1000	.99050	.10212
1001	.99050	.10198
1500	.99585	.05974
1501	.99585	.05968

NOTE : The maximum likelihood of plurality distortion in culture (ii) equals .43155 and occurs when N = 43.

Before doing this, it is readily verified that when N is very small and $p_i < 1/2$, $P_3(A_i ; N; p; (k+1)_L)$ contains the major portion of $P_3(A_i; N; p)$. By (7.34), as N increases $P_3(A_i; N; p; (k+1)_L)$ tends to zero. In other words, as N increases $P_3(A_i; N; p; k_U)$ requires a larger and larger share of $P_3(A_i; N; p)$ until $P_3(A_i; N; p; k_U) = P_3(A_i; N; p)$.

Consider culture (i), Table 7.3. From the previous paragraph it is apparent that $P_3(A_1; N; p; k_U)$ and $P_3(A_3; N; p; k_U)$ increase with N. Therefore, by (7.38), $PD_3(N; q)$ also becomes larger as N increases. Simultaneously, because p_3 is the largest first preference probability, $P_3(A_3; N; p)$ tends to unity as N becomes large. The net effect is that $PD_3(N; q)$ increases monotonically with odd N toward an asymptote of unity.

In culture (ii), Table 7.3, $P_3(A_1; N; p; k_U)$ and $P_3(A_3; N; p; k_U)$ become larger as N increases, for the reasons mentioned above. Using the normal approximation to the binomial distribution it is readily established that if $p_i \doteq 1/3$ then, by around $N = 45$, $P_3(A_i; N; p; k_U) \doteq P_3(A_i; N; p)$. However, simultaneously but more slowly $P_3(A_1; N; p)$ and $P_3(A_3; N; p)$ are shrinking because $P_3(A_2; N; p)$, being associated with the largest first preference probability, tends to unity as N increases. By (7.38) the net effect is an initial increase in $PD_3(N; q)$ which continues until $P_3(A_1; N; p; (k+1)_L)$ and $P_3(A_3; N; p; (k+1)_L)$ have been drained, i.e. at around $N = 45$, whereupon $PD_3(N; q)$ begins a gradual decline toward zero. More precisely, Table 7.3 reveals that the maximum likelihood of plurality distortion in culture (ii) is .43155 and occurs when $N = 43$.

Thus, we have established that two highly similar cultures can yield similar values for the likelihood of plurality distortion when N is small ($N \leq 20$) but markedly different values (unity and zero) when N becomes large. Moreover, the manner in which $PD_3(N; q)$ behaves as a function of N in these two highly similar cultures is also quite different: in one case increasing more or less monotonically with N to an asymptote of unity, and in the other case increasing with N to reach a maximum at N = 43 and then decreasing to an asymptote of zero. Lastly, the rate of approach by $PD_3(N; q)$ to the asymptote in the two cultures also seems at variance with their considerable similarity, culture (i) yielding a speedier approach than culture (ii).

Finally, the case of m = 4 alternatives is briefly examined. The asymptotic likelihood of plurality - Condorcet coincidence and of plurality distortion in a given culture for any value of m may be determined by means of expressions (7.30), (7.31), and (7.32). For more moderate values of N, formulae along the lines of (7.19) and (7.28) may be developed. However, for present purposes a computer search procedure, of the type used by Paris (1975) in case m = 3, was employed. To overcome the time limitations inherent in this approach the region over which the search takes place may be narrowed down substantially by means of inequalities (3.2).

Table 7.4 contains the likelihood of plurality - Condorcet coincidence and of plurality distortion in case m = 4 for groups of up to 20 members in each of three cultures. The cultures considered are : (i) the equiprobable culture, (ii) the culture in which the likelihood of the paradox of voting reaches its maximum and (iii) a single - peaked culture.

A comparison of Tables 7.2 and 7.4 reveals a general decrease in the likelihood of plurality - Condorcet coincidence in

TABLE 7.4

PROBABILITY OF PLURALITY DISTORTION (PD) AND OF PLURALITY - CONDORCET COINCIDENCE (PCC) FOR N GROUP MEMBERS AND $m = 4$ ALTERNATIVES IN EACH OF THREE CULTURES :

(i) $r_i = 1/24$, $i = 1, 2, ..., 24$;

(ii) $r_1 = r_{24} = 1/4$, $r_7 = r_9 = 1/8$, $r_{15} = r_{16} = r_{18} = 1/12$, $r_i = 0$, all other i ; and

(iii) $r_1 = r_{10} = r_{17} = r_{19} = 1/4$, $r_i = 0$, all other i.

	Culture (i)		Culture (ii)		Culture (iii)	
N	PD	PCC	PD	PCC	PD	PCC
3	.00000	.62500	.00000	.62500	.00000	.62500
4	.00000	.32813	.00000	.20313	.00000	.39063
5	.06727	.52257	.00000	.41406	.11719	.53125
6	.00000	.32834	.00000	.15039	.00000	.32617
7	.10324	.54997	.00000	.28223	.25635	.48730
8	.02293	.35288	.00000	.10919	.10254	.34845
9	.10934	.51702	.00000	.19571	.27878	.43604
10	.02733	.36091	.00000	.07891	.16422	.32805
11	.13552	.52983	.00000	.13731	.35028	.42150
12	.04421	.37328	.00000	.05701	.21810	.33503
13	.13695	.51675	.00000	.09716	.36049	.39787
14	.04943	.37995	.00000	.04124	.25999	.32691
15	.15353	.52313	.00000	.06920	.39960	.39262
16	.06183	.38823	.00000	.02988	.28948	.33199
17	.15491	.51651	.00000	.04954	.40444	.38121
18	.06648	.39322	.00000	.02169	.31527	.32838
19	.16636	.52045	.00000	.03561	.42848	.38018
20	.07631	.39946	.00000	.01577	.33248	.33301

corresponding cultures as m increases. This is not unexpected as the cultures in both Tables embody first preference equiprobability. As m increases the probability that a given alternative becomes plurality winner is bound to decrease. Hence, the likelihood of plurality - Condorcet coincidence also decreases.

Lastly, Table 7.5 contains two highly similar single - peaked cultures which display quite different behaviour, for the same reasons as the two cultures in case m = 3 presented in Table 7.3.

7.4 CULTURE IN WHICH THE LIKELIHOOD OF PLURALITY - CONDORCET DISAGREEMENT REACHES ITS MAXIMUM

The culture most likely to produce plurality - Condorcet disagreement is defined as that culture in which the probability of plurality - Condorcet disagreement, $PCD_m(N; q)$, reaches a given value close to unity, say .99999, with a smaller value of N than is required by any other culture.

Let $q_{max}(A_i; m)$ denote the culture where plurality - Condorcet disagreement is most likely when A_i is the asymptotic Condorcet winner and there are m alternatives.

Consider alternative A_2 in case m = 3. If A_2 is the asymptotic Condorcet winner then plurality - Condorcet disagreement will inevitably occur if either A_1 or A_3 is the asymptotic plurality winner. Accordingly, for the probability of plurality - Condorcet disagreement to reach the asymptote with the smallest possible value of N it is required that each of the following inequalities hold by as large a margin as possible :

TABLE 7.5

PROBABILITY OF PLURALITY DISTORTION FOR N GROUP MEMBERS AND m = 4 ALTERNATIVES IN TWO SINGLE - PEAKED CULTURES WITH FIRST PREFERENCE PROBABILITIES WHICH DEPART MARGINALLY FROM EQUIPROBABILITY. THE CULTURES, BOTH OF WHICH HAVE A_2 AND A_3 AS CENTRAL ALTERNATIVES, ARE :

(i) p = (11/40, 10/40, 9/40, 10/40), and

(ii) p = (10/40, 11/40, 10/40, 9/40).

N	Culture (i)	Culture (ii)
3	.00000	.00000
4	.00000	.00000
5	.12182	.11021
6	.00000	.00000
7	.27003	.23960
8	.10631	.09621
9	.29975	.25511
10	.17465	.15175
11	.37851	.31829
12	.23524	.19814
13	.39568	.32309
14	.28447	.23352
15	.44053	.35569
16	.32071	.25678
17	.45122	.35636
18	.35286	.27715
19	.47986	.37532
20	.37587	.28956
50	.51633	.34568
51	.58641	.39974
100	.59355	.35099
101	.64464	.38788
500	.78437	.27153
501	.80144	.28127
1000	.86830	.19224
1001	.87519	.19573

NOTE: The maximum likelihood of plurality distortion in culture (ii) equals .39987 and occurs when N = 47

$$q_1 + q_3 + q_4 > q_2 + q_5 + q_6$$
$$q_3 + q_4 + q_6 > q_1 + q_2 + q_5$$
$$p_1 > p_2 \qquad (7.40)$$
$$p_3 > p_2$$

It is evident that the margins of inequality are greatest when $q_2 = 0$, $q_5 = 0$, and $q_1 = q_6$. In this event let $Q = q_1 = q_6 = p_1 = p_3$. It follows that $1 - 2Q = p_2 = q_3 + q_4$. The first two inequalities in (7.40) reduce to the single inequality $1 - Q > Q$, or

$$Q < 1/2 \qquad (7.41)$$

Also, the third and fourth inequalities in (7.40) may both be rephrased as

$Q > 1 - 2Q$, or

$$Q > 1/3 \qquad (7.42)$$

Thus, the culture in which plurality - Condorcet disagreement is most likely to occur when A_2 is the asymptotic Condorcet winner is

$$q_{max}(A_2; 3) = (Q, 0, q_3, q_4, 0, Q) \qquad (7.43)$$

where $1/3 < Q < 1/2$, and $q_3 + q_4 = 1 - 2Q$. This culture has the property of single - peakedness.

It is possible to specify $q_{max}(A_2; 3)$ still more precisely. In $q_{max}(A_2; 3)$ it was found above that the first two inequalities in (7.40) reduce to the same inequality $Q < 1/2$. In other words, $q_{21} = q_{23} = 1 - Q > 1/2$. Thus, p($A_2$ simple majority over A_1) = p(A_2 simple majority over A_3). Now, in the asymptote p(A_2 simple majority over A_1) = 1.

Hence, substituting expression (4.30) we obtain the following expression for the probability that A_2 is Condorcet winner when N is large :

$C_3(A_2; N; q) = p(A_2$ simple majority over $A_1)$

$$= \sum_{V=k+1}^{N} \frac{N!}{V!(N-V)!} (1 - Q)^V Q^{N-V} \tag{7.44}$$

where $2k + 1 = N$ if N is odd and $2k = N$ if N is even, and V is the number of voters preferring A_2 to A_1. Employing the normal approximation to the binomial distribution given by (3.4) and rearranging terms, an expression may be derived for the minimum value of N necessary for A_2 to have probability $\Phi(z)$ of being Condorcet winner, where $\Phi(z)$ represents the probability under the normal distribution of obtaining a standard normal deviate of z or greater. Thus, approximately

$$N \geq \frac{z^2(1-Q)Q}{(Q-1/2)^2} \tag{7.45}$$

Now, the probability that either A_1 or A_3 is the plurality winner equals the probability that A_2 is a plurality loser. Also, we have established in $q_{max}(A_2; 3)$ that $p_2 = 1 - 2Q < 1/3$. Therefore, by expression (3.13) we find that as N becomes large

$p(A_1$ or A_3 is plurality winner)

$= p(A_2$ is plurality loser)

$$= \sum_{a_2=0}^{N/3} \frac{N!}{a_2!(N-a_2)!} (1 - 2Q)^{a_2} (2Q)^{N-a_2} \tag{7.46}$$

Once again, employing the normal approximation to the binomial distribution given by (3.4) and rearranging terms, an expression is obtained for the minimum value of N necessary for either A_1 or A_3 to have probability $\Phi(z)$ of being plurality winner. That is, approximately

$$N \geq \frac{z^2(1-2Q)2Q}{(2/3-2Q)^2} \tag{7.47}$$

In the asymptote (z) = 1 so that the values of z in expressions (7.45) and (7.47) are identical. Also, in the asymptote the value of Q must be such that expressions (7.45) and (7.47) give the same value for N. Therefore, we may equate the right hand sides of (7.45) and (7.47) in order to solve for Q :

$$\frac{z^2(1-Q)Q}{(Q-1/2)^2} = \frac{z^2(1-2Q)2Q}{(2/3-2Q)^2} \tag{7.48}$$

After simplification we obtain the quadratic equation

$$12Q^2 - 2Q - 1 = 0 \tag{7.49}$$

The solution of equation (7.49) yields Q = .3838. Therefore, the culture in which plurality - Condorcet disagreement is most likely to occur when A_2 is the asymptotic Condorcet winner is

$$q_{max}(A_2;\ 3) = (.3838,\ 0,\ q_3,\ q_4,\ 0,\ .3838) \tag{7.50}$$

where $q_3 + q_4 = .2324$.

When m = 4 there are 4! = 24 possible preference orderings (enumerated in chapter 2) with associated probabilities r_i, i = 1, 2, ..., 24. Following the same line of reasoning as in case m = 3, we find

$$q_{max}(A_2;\ 4) = \begin{cases} r_1 + r_2 = Q \\ r_{15} + r_{16} = Q \\ r_{21} + r_{22} = Q \\ p_2 = r_7 + r_8 + r_9 + r_{10} + r_{11} + r_{12} \\ \quad = 1 - 3Q \\ r_i = 0, \text{ all other } i. \end{cases} \tag{7.51}$$

where Q may be found by solving the asymptotic equation

$$\frac{(1-Q)Q}{(Q-1/2)^2} = \frac{3Q(1-3Q)}{(3/4-3Q)^2} \qquad (7.52)$$

Expression (7.52) simplifies to the quadratic equation

$$24Q^3 + 3Q - 3 = 0 \qquad (7.53)$$

the solution of which gives Q = .2965.

In general, $q_{max}(A_j; m)$ may be obtained as follows :

(i) the preference orderings in which A_j comes first have probabilities which sum to 1 - (m-1)Q ;

(ii) the preference orderings in which a given A_i comes first and A_j comes second have probabilities which sum to Q, for all $i \neq j$;

(iii) the remaining preference orderings each have zero probability;

(iv) Q is found from the asymptotic equality

$$\frac{(1-Q)Q}{(Q-1/2)^2} = \frac{(m-1)Q(1-(m-1)Q)}{((m-1)/m - (m-1)Q)^2} \qquad (7.54)$$

When m > 3 the culture $q_{max}(A_j; m)$ is no longer single - peaked, though it is still value restricted (Sen, 1970).

By definition $q_{max}(A_j; m)$ is the culture in which the probability of plurality - Condorcet disagreement reaches a given value close to unity with a smaller value of N than is required by any other culture. Because all the manipulations used above in the derivation of $q_{max}(A_j; m)$ depend heavily on asymptotic relations of one form or another, it might be expected that $q_{max}(A_j; m)$ need not necessarily be the culture in which plurality-Condorcet disagreement is most likely when N is small. However, in culture $q_{max}(A_j; m)$ it turns out that the likelihood of plurality - Condorcet disagreement reaches its maximum for all values of N.

How long does it take $PCD_m(N; q)$ to reach the asymptote when $q = q_{max}(A_j; m)$? In fact, the asymptote is reached surprisingly quickly. When $m = 3$ the probability of plurality-Condorcet disagreement attains a value of .99 around $N = 135$. When $m = 4$ the probability of .99 is gained around $N = 50$. Clearly, when $m > 4$ the asymptote will be reached when N is rather small.

CHAPTER 8

THE BORDA OUTCOME AND THE WILL OF THE MAJORITY

8.1 BORDA - CONDORCET COINCIDENCE

The Borda outcome may be said to reflect the will of the majority when it coincides with the Condorcet outcome, or, more precisely, when the same alternative emerges as both Borda and Condorcet winner. This event is referred to as Borda - Condorcet coincidence. Both procedures consider each group member's entire preference ordering of the alternatives. However, because the Borda procedure assigns a proportionally greater weight to an alternative the higher it stands in a member's preference ordering, it may produce a different winner from the one arrived at by the Condorcet procedure. This occurrence is termed Borda distortion. Borda - Condorcet coincidence and Borda distortion are thus analogues of plurality - Condorcet coincidence and plurality distortion.

In this chapter expressions are developed for the likelihood of Borda - Condorcet coincidence and the likelihood of Borda distortion. A solution for the former is established in the present section, and one for the latter in section 8.2.

A preliminary investigation of the likelihood of Borda - Condorcet coincidence was carried out by Fishburn (1974b). Using a computer simulation routine in which large numbers of preference orderings were randomly generated he obtained estimates of the likelihood of Borda - Condorcet coincidence in the equiprobable culture. Group sizes N = 5, 11, and 21 were examined in conjunction with m = 4, 5, 6, and 7 alternatives. Collapsing his results over m, Fishburn found that as group size increases the likelihood of Borda - Condorcet coincidence decreases, being .72 when N = 5, .68 when N = 11, and .65 when N = 21. These figures are undoubtedly overestimates because Fishburn included as instances of Borda - Condorcet coincidence situations in which the Condorcet winner is involved in a Borda winners' tie.

Let $BCC_m(N; q)$ represent the probability of Borda - Condorcet coincidence when N group members vote on m alternatives in culture q. If $BCC_m(A_i; N; q)$ denotes the probability that A_i is both the Borda and the Condorcet winner then we have

$$BCC_m(N; q) = \sum_{i=1}^{m} BCC_m(A_i; N; q) \qquad (8.1)$$

An expression for $BCC_m(A_i; N; q)$ is derived in case m = 3.

Consider alternative A_1. For A_1 to be both the Borda and the Condorcet winner the following sets of restrictions must hold :

$$B_1 > B_2$$

$$\text{(i.e. } x_1 + 2x_2 + x_5 > x_3 + 2x_4 + x_6 \text{)} \qquad (8.2)$$

$$B_1 > B_3$$

$$\text{(i.e. } 2x_1 + x_2 + x_3 > x_4 + x_5 + 2x_6 \text{)}$$

and

$$x_1 + x_2 + x_5 > x_3 + x_4 + x_6$$
$$x_1 + x_2 + x_3 > x_4 + x_5 + x_6 \quad (8.3)$$

Now, in chapter 5, from the inequalities (8.2) we arrived at the limits of summation (5.3) used by expression (5.2) to provide the probability that A_1 is the Borda winner. By modifying the limits of summation (5.3) to accomodate the additional inequalities (8.3) a solution for the probability that A_1 is both the Borda and the Condorcet winner may be obtained.

It will be recalled that for given a_i, b_i, and c_i, $i = 1, 2, 3$, knowledge of the value of any one of the x_j enables the values of every x_j, $j = 1, 2, ..., 6$, to be determined. It is convenient to employ the variable of summation x_3 in place of x_1 in (5.3). Altering the limits of summation accordingly, we obtain the following equivalent form of (5.3) :

$$
\begin{aligned}
N+1 &\le B_1 \le 2N \\
\max[3N-2B_1+1,0] &\le B_2 \le \min[B_1-1,\ 3N-B_1] \\
B_1-N &\le a_1 \le \min[B_1/2, 2N-B_2, B_1+B_2-N] \\
\max[\{(B_1+B_2-2a_1-N+1)/2\}, B_2-N, 0] &\le a_2 \le \min[B_2/2, N-a_1, B_1+B_2-N-a_1] \\
\max[a_2-b_3, a_2-c_1, 0] &\le x_3 \le \min[a_2, b_1, c_3]
\end{aligned}
\quad (8.4)
$$

When inequalities (8.3) hold in addition to inequalities (8.2), several modifications to the limits in (8.4) are required. Now, B_1 attains its lower limit when $3N-B_1 = B_2+B_3$ reaches its upper limit. From (8.3) $a_2 = x_3 + x_4$ and $a_3 = x_5 + x_6$ both have an upper limit of k, where $2k+1 = N$ if N is odd and $2k+2 = N$ if N is even. Therefore, the upper limit of $b_2 + b_3 = N-2k$. Hence, the upper limit of

$B_2 + B_3 = 4k + N-2k = 2k+N$, thereby giving B_1 a lower limit of $3N - (2k+N)$ or $2N - 2k$.

The lower limit of a_1 is attained when $N - a_1 = a_2 + a_3$ assumes its upper limit. From (8.3) and (8.4) a_2's upper limit equals k when $a_1 \leq N - k$ and $B_2 \geq 2k$. In this event $a_1 + a_3 = N-k$. Now, when N is even and $B_2 = B_1 - 1$ then $B_3 = 3N - 2B_1 + 1$ is always odd. Therefore, the upper limit of $a_3 = (3N-2B_1)/2 = N + k + 1 - B_1$, giving $a_2 + a_3$ an upper limit of $N + 2k + 1 - B_1$. Hence, the lower limit of $a_1 = N - (N + 2k + 1 - B_1) = B_1 - 1 - 2k = B_2 - 2k$. Similarly, when N is even and $B_3 = B_1 - 1$ then a_1 has a lower limit of $B_3 - 2k$, or equivalently $3N - B_1 - B_2 - 2k$. Otherwise, a_1's lower limit remains $B_1 - N$.

We have already noted that the inequalities (8.3) require a_2 to be less than or equal to k. They also require that $a_3 \leq k$, or $N - a_1 - a_2 \leq k$, or $a_2 \geq N - k - a_1$.

Lastly, by (8.3), $x_3 + x_4 + x_6 \leq k$, or $x_3 \leq k - x_4 - x_6$, or $x_3 \leq k - c_1$; and $x_1 + x_2 + x_3 \geq N - k$, or $x_3 \geq N - k - a_1$.

The foregoing modifications to the limits in (8.4) enable us to obtain an expression for the probability that A_1 is both the Borda and the Condorcet winner. Thus,

$$BCC_3(A_1;\ N;\ q) = \sum^{5} \frac{N!}{x_1!x_2!\ldots\ldots x_6!}\, q_1^{x_1}\, q_2^{x_2} \ldots\ldots q_6^{x_6} \qquad (8.5)$$

where $\sum^{5}$ is a 5-fold summation whose variables of summation have the following limits

$$2N-2k \le B_1 \le 2N$$

$$\max\left[3N-2B_1+1, 0\right] \le B_2 \le \min\left[B_1-1, 3N-B_1\right]$$

$$\max\left[B_2-2k, 3N-B_1-B_2-2k, B_1-N\right] \le a_1 \le \min\left[B_1/2, 2N-B_2, B_1+B_2-N\right] \qquad (8.6)$$

$$\max\left[\left\{(B_1+B_2-2a_1-N+1)/2\right\}, N-k-a_1, B_2-N, 0\right] \le a_2 \le \min\left[k, B_2/2, N-a_1, B_1+B_2-N-a_1\right]$$

$$\max\left[N-k-a_1, a_2-b_3, a_2-c_1, 0\right] \le x_3 \le \min\left[a_2, b_1, c_3, k-c_1\right]$$

and where $x_1 = c_3 - x_3$, $x_2 = a_1 - x_1$, $x_4 = a_2 - x_3$, $x_5 = b_1 - x_3$, and $x_6 = b_2 - x_1$. Similar expressions may be obtained for $BCC_3(A_2; N; q)$ and $BCC_3(A_3; N; q)$.

Finally, by (8.1) and (8.5) we have an expression for the probability of Borda - Condorcet coincidence :

$$BCC_3(N; q)$$

$$= \sum_{i=1}^{3} BCC_3(A_i; N; q)$$

$$= \sum^{5} \frac{N!}{x_1!x_2!x_3!x_4!x_5!x_6!} \left[q_1^{x_1} q_2^{x_2} q_3^{x_3} q_4^{x_4} q_5^{x_5} q_6^{x_6} + q_1^{x_3} q_2^{x_4} q_3^{x_1} q_4^{x_2} q_5^{x_6} q_6^{x_5} + q_1^{x_6} q_2^{x_5} q_3^{x_4} q_4^{x_3} q_5^{x_2} q_6^{x_1} \right] \qquad (8.7)$$

where $\sum^{5}$ is a 5-fold summation whose variables of summation have limits given by (8.6). Note that $BCC_3(A_i; N; q)$, $i = 1, 2, 3$, have been brought together under a single 5-fold summation sign. This is achieved by interchanging the labels for the alternatives to establish corresponding preference orderings, in a manner similar to that described in connection with expression (7.19).

8.2 BORDA DISTORTION

Let $BD_m(N; q)$ represent the probability of Borda distortion when N group members vote on m alternatives in culture q. That is, $BD_m(N; q)$ is the probability that the Borda method selects a non-Condorcet winner when a Condorcet winner is available. If $BD_m(A_i, A_j; N; q)$ denotes the probability that A_i is the Borda winner while A_j is the Condorcet winner then we have

$$BD_m(N; q) = \sum_{\substack{i,j=1 \\ i \neq j}}^{m} BD_m(A_i, A_j; N; q) \tag{8.8}$$

An expression for $BD_m(A_i, A_j; N; q)$ is developed in case m = 3.

Consider the situation in which A_1 is the Borda winner and A_2 is the Condorcet winner. For this situation to arise the following sets of inequalities must hold :

$$B_1 > B_2$$

$$\text{(i.e. } x_1 + 2x_2 + x_5 > x_3 + 2x_4 + x_6 \text{)} \tag{8.9a}$$

$$B_1 > B_3$$

$$\text{(i.e. } 2x_1 + x_2 + x_3 > x_4 + x_5 + 2x_6 \text{)} \tag{8.9b}$$

and

$$x_3 + x_4 + x_6 > x_1 + x_2 + x_5 \tag{8.10a}$$

$$x_1 + x_3 + x_4 > x_2 + x_5 + x_6 \tag{8.10b}$$

It is interesting to note that (8.9a), (8.10a) and (8.10b) imply that $B_2 > B_3$. Therefore, (8.9a), (8.10a) and (8.10b) must also imply inequality (8.9b). To prove this we observe that (8.10a) and (8.10b) imply that

$$x_3 + x_4 > x_2 + x_5 \quad (8.11)$$

Inequalities (8.9a) and (8.10a) imply that

$$x_2 > x_4 \quad (8.12)$$

From (8.11) and (8.12) we have

$$x_3 > x_5 , \quad (8.13)$$

which taken in conjunction with (8.10b) gives

$$x_1 + 2x_3 + x_4 > x_2 + 2x_5 + x_6$$
$$(\text{i.e. } B_2 > B_3) \quad (8.14)$$

Since Borda scores behave transitively we have $B_1 > B_3$. In other words, if A_j is the Condorcet winner and A_i receives a higher Borda score than A_j then A_i must be the Borda winner. This contrasts with the plurality procedure in which it is possible for a plurality loser, as well as the plurality winner, to defeat the Condorcet winner.

Proceeding as in the case of Borda - Condorcet coincidence, we make use of the fact that inequalities (8.9a) and 8.9b) were shown in chapter 5 to yield the region of summation (5.3). By modifying (5.3) to accomodate the additional inequalities (8.10a) and (8.10b) a solution for the probability that A_1 is the Borda winner while A_2 is the Condorcet winner may be obtained.

From (8.10a) it is apparent that $a_1 = x_1 + x_2 \leq k$, where $2k + 1 = N$ if N is odd and $2k + 2 = N$ if N is even. Since $k \leq \min\left[B_1/2,\ 2N - B_2,\ B_1 + B_2 - N\right]$ the upper bound of a_1 becomes simply k. The upper limit of B_1 is reached when a_1 and b_1 are at their largest, i.e. k and N - k respectively. Therefore, $B_1 \leq 2k + N - k$, or $B_1 \leq N + k$. From (8.14) it is clear that B_1's lower limit equals B_2's lower limit plus unity. Now, B_2 attains its lower limit when $3N - B_2 = B_1 + B_3$ reaches its upper limit. From (8.10a) and (8.10b) $a_1 = x_1 + x_2$ and

$a_3 = x_5 + x_6$ both have an upper limit of k. Thus, the upper limit of $b_1 + b_3$ equals N - 2k. Hence, the upper limit of $B_1 + B_3 = 4k + N - 2k = 2k + N$, thereby fixing B_2's lower limit at 3N - (2k + N) or 2N - 2k. Since $2N - 2k > \max[3N - 2B_1 + 1, 0]$ the former replaces the latter as B_2's lower limit. Therefore, B_1's lower limit equals 2N - 2k + 1. Because $B_1 \geq N + k$, the upper limit of B_2 becomes simply $B_1 - 1$.

Because $B_2 \geq 2N - 2k$ and $a_1 \leq k$ it follows that $B_1 + B_2 - N - a_1 > B_2/2$. Hence, $B_1 + B_2 - N - a_1$ no longer forms part of the upper limit of a_2. For given B_1, B_2, and a_1, from (8.10a) we have $x_3 + x_4 + x_6 \geq N - k$, or $x_3 + x_4 + N - a_1 - b_1 - x_4 \geq N - k$. Therefore $x_3 \geq a_1 + b_1 - k$, or $x_3 \geq B_1 - a_1 - k$. When $x_3 = B_1 - a_1 - k$ then $x_4 = 0$. Hence, a_2 shares the same lower limit as x_3, i.e. $a_2 \geq B_1 - a_1 - k$. From (8.10b) we have $x_1 + x_3 + x_4 \geq N - k$, or $x_1 \geq N - k - a_2$, which goes to form part of x_1's lower limit. Lastly, by (8.10a) $x_1 + x_2 + x_5 \leq k$ or $x_1 \leq k - c_2$.

The above modifications to the limits in (5.3) allow us to obtain an expression for the probability that A_1 is the Borda winner while A_2 is the Condorcet winner. Thus,

$$BD_3(A_1, A_2; N; q) = \overset{5}{\Sigma} \frac{N!}{x_1!x_2!\ldots\ldots x_6!} q_1^{x_1} q_2^{x_2} \ldots\ldots q_6^{x_6} \qquad (8.15)$$

where $\overset{5}{\Sigma}$ is a 5-fold summation whose variables of summation have the following limits

$$
\begin{aligned}
2N-2k+1 &\leq B_1 \leq N+k \\
2N-2k &\leq B_2 \leq B_1-1 \\
B_1-N &\leq a_1 \leq k \\
B_1-a_1-k &\leq a_2 \leq \min\left[B_2/2, N-a_1\right] \\
\max\left[N-k-a_2, a_1-c_2, a_1-b_3, 0\right] &\leq x_1 \leq \min\left[k-c_2, a_1, b_2, c_3\right]
\end{aligned} \tag{8.16}
$$

Similar expressions may be obtained for $BD_3(A_1, A_3; N; q)$, $BD_3(A_2, A_1; N; q)$, $BD_3(A_2, A_3; N; q)$, $BD_3(A_3, A_1; N; q)$ and $BD_3(A_3, A_2; N; q)$.

Finally, by (8.8) we arrive at an expression for the probability of **Borda** distortion :

$$
\begin{aligned}
BD_3(N; q) = \sum^{5} \frac{N!}{x_1!x_2!x_3!x_4!x_5!x_6!} \Big[& q_1^{x_1} q_2^{x_2} q_3^{x_3} q_4^{x_4} q_5^{x_5} q_6^{x_6} \\
& + q_1^{x_2} q_2^{x_1} q_3^{x_5} q_4^{x_6} q_5^{x_3} q_6^{x_4} + q_1^{x_3} q_2^{x_4} q_3^{x_1} q_4^{x_2} q_5^{x_6} q_6^{x_5} \\
& + q_1^{x_5} q_2^{x_6} q_3^{x_2} q_4^{x_1} q_5^{x_4} q_6^{x_3} + q_1^{x_4} q_2^{x_3} q_3^{x_6} q_4^{x_5} q_5^{x_1} q_6^{x_2} \\
& + q_1^{x_6} q_2^{x_5} q_3^{x_4} q_4^{x_3} q_5^{x_2} q_6^{x_1} \Big]
\end{aligned} \tag{8.17}
$$

where $\sum^{5}$ is a 5-fold summation whose variables of summation have limits given by (8.16). As in previous expressions of this kind it has been possible to bring the constituent summations under a single 5-fold summation sign simply by interchanging the labels for the alternatives in order to establish corresponding preference orderings.

8.3 EFFECT OF GROUP SIZE AND CULTURE ON THE LIKELIHOOD OF AGREEMENT BETWEEN BORDA AND CONDORCET OUTCOMES

By means of formulae (8.7) and (8.17) the likelihoods of

Borda-Condorcet coincidence and Borda distortion, in case $m = 3$, were calculated in each of six cultures for group sizes $N = 3, 4, \ldots, 100$. The resultant probability values are contained in Table 8.1 in the case of Borda - Condorcet coincidence, and in Table 8.2 in the case of Borda distortion. For comparative purposes, also included in Tables 8.1 and 8.2 are values for the likelihoods of plurality - Condorcet coincidence and plurality distortion, respectively. Culture (i) is the equiprobable culture; cultures (ii) (iii), and (vi) have the property of single - peakedness ; culture (iv) is the culture in which the likelihood of the paradox of voting is at its maximum; and culture (v) is a culture in which the plurality and Condorcet outcomes always agree.

When $N = 4$ the probability of Borda - Condorcet coincidence equals that of plurality - Condorcet coincidence. When $N = 3$, 4, and 6 the probability of Borda distortion equals zero. The differences between neighbouring odd-sized and even-sized groups are caused by the higher probability of a tie in even-sized groups under both the Borda and the Condorcet procedures.

Fishburn (1974b) concluded from his computer simulation study that the probability of Borda - Condorcet coincidence in the equiprobable culture decreases as N increases. From Table 8.1 it is evident that this conclusion is erroneous. In fact, the likelihood of Borda - Condorcet coincidence in culture (i) increases with N when N is even and remains approximately constant at .8 when N is odd.

Contrasting the plurality and Borda methods in terms of the degree to which each accords with the will of the majority, Black (1958, p.177) agrees with Condorcet that "Borda's method would lead to a wrong (i.e. non-Condorcet) result in only a small number of cases. It is much superior to the single vote (i.e.plurality method) ...." In other words, Black and Condorcet share the opinion that Borda - Condorcet coincidence has a substantially higher likelihood of occurrence than plurality - Condorcet coincidence, and correspondingly that the likelihood of Borda distortion is much lower than that of plurality distortion. Now, this assertion is incomplete as it stands since neither Black nor Condorcet specify the cultures to which their remarks apply. Clearly, the assertion does not hold for all cultures. From Tables 8.1 and 8.2 it is apparent that in culture (v) and in culture (vi) the plurality outcome is considerably more likely to correspond to the will of the majority than is the Borda outcome. In fact, in culture (v) plurality distortion cannot occur, whereas the likelihood of Borda distortion increases with N to a limiting value of unity. (The likelihood of plurality - Condorcet coincidence in this culture does not equal unity when N is even because of the possibility of ties.) In general, from the discussion in chapter 7 of the asymptotic behaviour of the likelihood of plurality distortion, it is evident that as N becomes large plurality distortion is exceedingly unlikely to occur in cultures where $p_i > 1/2$, some i. By contrast, as we have just seen, Borda distortion may have a sizeable probability of arising in such cultures.

On the other hand, in culture (i) and in culture (iii) the Borda outcome is consistently more likely to correspond to the will of the majority than is the plurality outcome. Thus, there exist cultures which are consonant with the view expressed by Black and Condorcet as well as cultures which conflict with that view. Whether or not any one of these cultures is more "typical" or "realistic" than the others is an, as yet unanswered, empirical question.

Finally, it is interesting to observe in case m = 3 that the Condorcet winner may receive the least number of first preference votes, thereby occupying last place in the plurality social ordering. However, as will be demonstrated in the next chapter, the Condorcet winner cannot occupy last place in the Borda social ordering. Thus, distortions of the will of the majority introduced by the Borda procedure would appear to be of a potentially less extreme form than those introduced by the plurality procedure.

TABLE 8.1

THE COMPARATIVE LIKELIHOOD OF BORDA - CONDORCET COINCIDENCE (BCC) AND PLURALITY - CONDORCET COINCIDENCE (PCC) FOR N GROUP MEMBERS AND m = 3 ALTERNATIVES IN EACH OF SIX CULTURES :

(i) q = (1/6, 1/6, 1/6, 1/6, 1/6, 1/6)

(ii) q = (1/4, 0, 1/4, 1/4, 0, 1/4), ... continued overleaf

	Culture (i)		Culture (ii)	
N	BCC	PCC	BCC	PCC
3	.86111	.77778	.90625	.81250
4	.44444	.44444	.60156	.60156
5	.82253	.62963	.90234	.70703
6	.50682	.48560	.73389	.65332
7	.80871	.68193	.91882	.74365
8	.54650	.48388	.81607	.65714
9	.80315	.68660	.93687	.75166
10	.57414	.51109	.87022	.71121
11	.80081	.65110	.95210	.74489
12	.59473	.52714	.90733	.73999
13	.79988	.67469	.96404	.77483
14	.61082	.52500	.93334	.75239
15	.79960	.67812	.97314	.79235
16	.62386	.54146	.95183	.78486
17	.79964	.65900	.98001	.80037
18	.63470	.55058	.96507	.80629
19	.79985	.67401	.98516	.82340
20	.64391	.54814	.97461	.81890
49	.80454	.67667	.99984	.95906
50	.70760	.59605	.99976	.96060

TABLE 8.1 (continued)

(iii) q = (1/2, 0, 1/4, 0, 0, 1/4),

(iv) q = (1/3, 0, 0, 1/3, 1/3, 0),.....continued overleaf

	Culture (iii)		Culture (iv)	
N	BCC	PCC	BCC	PCC
3	.95313	.81250	.44444	.77778
4	.50781	.50781	.33333	.33333
5	.95117	.70703	.50617	.62963
6	.59106	.50684	.23868	.30041
7	.95941	.69238	.47188	.51989
8	.65115	.48624	.23823	.26383
9	.96844	.65553	.32571	.43454
10	.69659	.49892	.21902	.22969
11	.97605	.61932	.31372	.36626
12	.73158	.49194	.16972	.19934
13	.98202	.61087	.28559	.31062
14	.75883	.47978	.15841	.17285
15	.98657	.59434	.21773	.26469
16	.78030	.48289	.14295	.14989
17	.99001	.57676	.20163	.22643
18	.79743	.47844	.11539	.13004
19	.99258	.57168	.18144	.19430
20	.81128	.47152	.10522	.11291
49	.99992	.51391	.02306	.02362
50	.88760	.45678	.01440	.01475

TABLE 8.1

(v) q = (0, 5/12, 7/12, 0, 0, 0),

(vi) q = (.1, .2, .6, 0, .1, 0).

	Culture (v)		Culture (vi)	
N	BCC	PCC	BCC	PCC
3	.57465	1.00000	.78100	.89200
4	.64554	.64554	.62740	.62740
5	.65539	1.00000	.73910	.85420
6	.41129	.71282	.63549	.68410
7	.70684	1.00000	.74522	.85938
8	.48209	.75570	.62471	.71508
9	.50407	1.00000	.73319	.85885
10	.53685	.78624	.62560	.74324
11	.54279	1.00000	.71215	.85390
12	.38086	.80950	.62520	.76186
13	.57692	1.00000	.69559	.86361
14	.42070	.82802	.61970	.77652
15	.43501	1.00000	.68289	.86709
16	.45700	.84324	.61302	.79009
17	.46149	1.00000	.67093	.87065
18	.33925	.85607	.60706	.80164
19	.48719	1.00000	.65941	.87483
20	.36715	.86706	.60144	.81201
49	.24639	1.00000	.56077	.93163
50	.19694	.94448	.54071	.91086

TABLE 8.2

THE COMPARATIVE LIKELIHOOD OF BORDA DISTORTION (BD) AND PLURALITY DISTORTION (PD) FOR N GROUP MEMBERS AND m = 3 ALTERNATIVES IN EACH OF SIX CULTURES :

(i) q = (1/6, 1/6, 1/6, 1/6, 1/6, 1/6),

(ii) q = (1/4, 0, 1/4, 1/4, 0, 1/4),.....continued overleaf

	Culture (i)		Culture (ii)	
N	BD	PD	BD	PD
3	.00000	.00000	.00000	.00000
4	.00000	.00000	.00000	.00000
5	.00772	.00000	.01953	.00000
6	.00000	.00000	.00000	.00000
7	.01725	.07202	.02991	.10254
8	.00020	.00000	.00171	.00000
9	.02536	.08102	.03172	.11536
10	.00081	.01667	.00360	.06409
11	.03177	.06999	.02902	.10134
12	.00178	.02323	.00482	.07931
13	.03683	.10703	.02461	.12458
14	.00297	.02369	.00523	.07518
15	.04089	.11033	.01999	.11948
16	.00430	.03891	.00505	.09451
17	.04421	.09937	.01581	.10397
18	.00569	.04363	.00454	.09353
19	.04698	.12434	.01228	.10775
20	.00710	.04256	.00388	.08396
49	.06463	.15775	.00016	.02750
50	.02348	.09217	.00007	.02552
99	.07272	.17523	.00000	.00251
100	.03727	.12724	.00000	.00240

TABLE 8.2 (continued)

(iii) q = (1/2, 0, 1/4, 0, 0, 1/4),

(iv) q = (1/3, 0, 0, 1/3, 1/3, 0).....continued overleaf

	Culture (iii)		Culture (iv)	
N	BD	PD	BD	PD
3	.00000	.00000	.00000	.00000
4	.00000	.00000	.00000	.00000
5	.00977	.00000	.12346	.00000
6	.00000	.00000	.00000	.00000
7	.01495	.15381	.04801	.00000
8	.00085	.00000	.02561	.00000
9	.01586	.21149	.01920	.00000
10	.00180	.09613	.01067	.00000
11	.01451	.22691	.05253	.00000
12	.00241	.14540	.00447	.00000
13	.01231	.28854	.02502	.00000
14	.00261	.16790	.01444	.00000
15	.01000	.31749	.01181	.00000
16	.00253	.22052	.00694	.00000
17	.00790	.32758	.02480	.00000
18	.00227	.25020	.00330	.00000
19	.00614	.35947	.01285	.00000
20	.00194	.26524	.00769	.00000
49	.00008	.47265	.00056	.00000
50	.00004	.41715	.00035	.00000
99	.00000	.49760	.00000	.00000
100	.00000	.45791	.00000	.00000

TABLE 8.2. (continued)

(v) q = (0, 5/12, 7/12, 0, 0, 0),

(vi) q = (.1, .2, .6, 0, .1, 0).

	Culture (v)		Culture (vi)	
N	BD	PD	BD	PD
3	.00000	.00000	.00000	.00000
4	.00000	.00000	.00000	.00000
5	.34461	.00000	.08650	.00000
6	.00000	.00000	.00000	.00000
7	.29316	.00000	.14524	.04763
8	.27361	.00000	.03484	.00000
9	.25651	.00000	.17249	.06246
10	.24939	.00000	.07838	.01715
11	.45721	.00000	.19415	.06945
12	.22861	.00000	.11352	.02473
13	.42308	.00000	.21676	.07985
14	.40731	.00000	.14204	.02979
15	.39293	.00000	.23704	.08503
16	.38625	.00000	.16779	.03635
17	.53851	.00000	.25412	.08784
18	.36601	.00000	.19120	.04089
19	.51281	.00000	.26922	.09026
20	.49991	.00000	.21197	.04433
49	.75361	.00000	.39440	.06387
50	.74755	.00000	.37445	.04448
99	.89713	.00000	.45831	.02121
100	.89569	.00000	.45324	.01608

CHAPTER 9

THE BORDA REVERSED - ORDER PARADOX

9.1 PROBABILITY OF A BORDA REVERSAL

In the Borda reversed-order paradox the removal of one of the m alternatives, after the voters have recorded their preference orderings, produces an inconsistency in the Borda outcomes whereby the social ordering on the reduced set of m - 1 alternatives conflicts with the social ordering on the full set of m alternatives.

For example, if N = 5 members of an appointments committee hold the following preference orderings in respect of m = 4 candidates for a post,

A_1	A_2	A_2	A_3	A_4
A_3	A_1	A_4	A_2	A_1
A_2	A_4	A_1	A_4	A_3
A_4	A_3	A_3	A_1	A_2 ,

the resultant Borda scores are

$B_1 = 3 + 2 + 1 + 0 + 2 = 8$

$B_2 = 1 + 3 + 3 + 2 + 0 = 9$

$B_3 = 2 + 0 + 0 + 3 + 1 = 6$

$B_4 = 0 + 1 + 2 + 1 + 3 = 7$.

Now, if candidate A_4 withdraws, the original preference orderings with A_4 removed become

A_1	A_2	A_2	A_3	A_1
A_3	A_1	A_1	A_2	A_3
A_2	A_3	A_3	A_1	A_2 ,

yielding the revised Borda scores

$B_1 = 2 + 1 + 1 + 0 + 2 = 6$

$B_2 = 0 + 2 + 2 + 1 + 0 = 5$

$B_3 = 1 + 0 + 0 + 2 + 1 = 4$.

Whereas A_2 is the winner when the full set of alternatives is considered, the winner of the reduced set is A_1. That is, using the Borda method, the committee initially selects A_2 but following the withdrawal of A_4 it selects A_1.

A similar inconsistency occurs in the above example if A_1 is removed from the full set of alternatives. Whereas $B_4 > B_3$ when all four candidates are considered, the withdrawal of A_1 results in the outcome $B_3 > B_4$. Moreover, if candidate A_2 also decides to withdraw so that only A_3 and A_4 remain, the Borda scores of the latter two candidates once again undergo a relative magnitude reversal, i.e. $B_4 > B_3$.

For a given pair of alternatives A_i and A_j with Borda scores B_i and B_j , a Borda reversal is said to occur if $B_i > B_j$ when the Borda scores are derived from the full set of m alternatives, but $B_j > B_i$ when the Borda scores are calculated on the basis of a reduced set of m - 1 alternatives.

Previous research has addressed itself to the investigation of specific subcategories of Borda reversal. Davidson and Odeh (1972) studied changes in the social ordering of the Borda losers following the removal of the original Borda winner. They examined the conditions under which an alternative, who received a lower Borda score than the winner and at least one other alternative, receives the highest Borda score when the winner is removed. Fishburn (1974b) proved that it is possible for the original social ordering of the Borda losers to be completely inverted when the original Borda winner is removed. He also cited a dual form of this paradox in which removal of the last - place candidate results in the complete reversal of the order of the first m - 1 alternatives. By means of a computer simulation procedure Fishburn (1974b) estimated the likelihood of occurrence of two types of Borda reversal: firstly, that investigated by Davidson and Odeh (1972) in which removal of the original winner results in the original second - place alternative becoming a Borda loser and, secondly, that in which the original winner becomes a loser when one of the original losers is removed. Group sizes N = 5, 11, and 21 were examined in conjunction with m = 4, 5, 6, and 7 alternatives. Only the equiprobable culture was considered. Fishburn established that the likelihood of occurrence of both types of Borda reversal increases as the number of group members increases. Thus, collapsing over m, the first type was found to have a probability of occurrence approximately equal to .02 when N = 5, .05 when N = 11, and .07 when N = 21.

As an example of the results obtained for the second type, which were not collapsed over m, a probability of occurrence was found in case m = 4 of approximately .04 when N = 5, .07 when N = 11, and .14 when N = 21.

In the present study specific subcategories of Borda reversal are not considered in depth. Rather, the principal focus of interest is the probability of occurrence of a Borda reversal involving any two alternatives. Attention is confined in the main to the case of m = 3 alternatives.

With a full set of m alternatives there are m possible reduced sets of m - 1 alternatives. Let $BR_m(N; q)$ represent the probability, for N group members and m alternatives in culture q, that at least one Borda reversal occurs in at least one of the m possible reduced sets of m - 1 alternatives. Also, let $BR_m(A_i; N; q)$ denote the probability that at least one Borda reversal occurs in the reduced set created by the removal of A_i. Similarly, $BR_m(A_i, A_j; N; q)$ denotes the probability that at least one Borda reversal occurs in both of the reduced sets of m - 1 alternatives created by the separate removals of A_i and A_j; $BR_m(A_i, A_j, A_k; N; q)$ denotes the probability that at least one Borda reversal occurs in all three of the reduced sets of m - 1 alternatives created by the separate removals of A_i, A_j and A_k ; ....; and $BR_m(A_1, A_2, ..., A_m; N; q)$ denotes the probability that at least one Borda reversal occurs in every one of the m possible reduced sets of m - 1 alternatives.

Using the well - known formula for the probability of the realisation of at least one among m events (Feller, 1968) we may write

$$BR_m(N;\ q) = \sum_{i=1}^{m} BR_m(A_i;\ N;\ q) - \sum_{i<j} BR_m(A_i,\ A_j;\ N;\ q)$$

$$+ \sum_{i<j<k} BR_m(A_i,\ A_j,\ A_k;\ N;\ q) - \ldots\ldots.$$

$$\ldots\ldots + (-1)^{m+1}\ BR_m(A_1,\ A_2,\ \ldots.,\ A_m;\ N;\ q) \quad (9.1)$$

Expressions for $BR_m(A_i;\ N;\ q)$ and $BR_m(A_i,\ A_j;\ N;\ q)$ are developed in the case of $m = 3$ alternatives, and it is shown that $BR_3(A_1,\ A_2,\ A_3;\ N;\ q) = 0$. In this way an exact solution for $BR_3(N;\ q)$ is obtained.

Consider alternative A_1 in the case of $m = 3$ alternatives. A Borda reversal following the removal of A_1 can take place in two ways depending on whether originally $B_2 > B_3$ or $B_3 > B_2$.

Consider the case $B_2 > B_3$. The inequality $B_2 > B_3$ may be written as

$$x_1 + 2x_3 + x_4 > x_2 + 2x_5 + x_6 \quad (9.2)$$

When A_1 is removed, for a Borda reversal to occur it is required that A_3 defeat A_2. As only two alternatives remain the Borda procedure here is equivalent to the Condorcet procedure. Hence, it is required that

$$x_2 + x_5 + x_6 > x_1 + x_3 + x_4 \quad (9.3)$$

Define $d_1 = x_1 + x_4$ and $d_2 = x_2 + x_6$. Inequalities (9.2) and (9.3) may be rewritten as

$$d_1 + 2x_3 > d_2 + 2x_5 \quad (9.4)$$

and

$$d_2 + x_5 > d_1 + x_3 \quad (9.5)$$

Together (9.4) and (9.5) define all the values of d_1, d_2, x_3 and x_5 which lead to a Borda reversal of $B_2 > B_3$ to $B_3 > B_2$

when A_1 is removed.

From (9.5) it is clear that $d_1 + x_3 \leq k$, where $2k + 1 = N$ if N is odd and $2k + 2 = N$ if N is even. From (9.4), for given d_1 the lowest value taken by x_3 occurs when $x_5 = 0$. Therefore, the upper limit of d_1 is attained when $d_1 + x_3 = k$ and $x_5 = 0$, i.e. when $d_1 + 2x_3 = N - k + 1$, or $x_3 + k = N - k + 1$, or $x_3 = N - 2k + 1$. Thus, the upper limit of d_1 equals $3k - N - 1$. The lower limit of d_1 equals zero. As mentioned previously, for given d_1 the lower limit of x_3 occurs when $x_5 = 0$, in which event $d_1 + 2x_3 \geq d_2 + 1$, or $d_1 + 2x_3 \geq N - d_1 - x_3 + 1$, or $x_3 \geq \{(N - 2d_1 + 3)/3\}$, where the special brackets signify that the integer value of the expression within is required. The upper limit of x_3 equals $k - d_1$. For given d_1 and x_3, the lower limit of x_5 equals zero. Lastly, the upper limit of x_5 is such that

$2x_5 \leq d_1 + 2x_3 - d_2 - 1$, or

$2x_5 \leq d_1 + 2x_3 - (N - d_1 - x_3 - x_5) - 1$, or

$x_5 \leq 2d_1 + 3x_3 - N - 1$.

When a Borda reversal following the removal of A_1 occurs in the second of the two possible ways, we have originally $B_3 > B_2$ but $B_2 > B_3$ on the withdrawal of A_1. In other words the following inequalities hold :

$$d_2 + 2x_5 > d_1 + 2x_3 \qquad (9.6)$$

and

$$d_1 + x_3 > d_2 + x_5 \qquad (9.7)$$

By symmetry, the limits of d_1, d_2, x_3, and x_5 as defined by (9.6) and (9.7) may be obtained from those determined above by interchanging the limits of d_1 with those of d_2 , and the limits of x_3 with those of x_5 .

The probability that a voter chooses either of the preference orderings s_1 or s_4 equals $q_1 + q_4$, and the probability that a voter chooses either s_2 or s_6 equals $q_2 + q_6$. Applying the above results to the multinomial social choice model we obtain an expression for the probability of a Borda reversal following the removal of A_1:

$$BR_3(A_1; N; q) = \sum^{3} \frac{N!}{d_1!x_3!x_5!d_2!}\left[(q_1 + q_4)^{d_1} q_3^{x_3} q_5^{x_5} (q_2 + q_6)^{d_2} + (q_1 + q_4)^{d_2} q_3^{x_5} q_5^{x_3} (q_2 + q_6)^{d_1}\right] \tag{9.8}$$

where $\sum^{3}$ is a 3 - fold summation whose variables of summation have the following limits

$$\begin{aligned} 0 &\leq d_1 \leq 3k - N - 1 \\ \left\{(N - 2d_1 + 3)/3\right\} &\leq x_3 \leq k - d_1 \\ 0 &\leq x_5 \leq 2d_1 + 3x_3 - N - 1 \end{aligned} \tag{9.9}$$

and where $d_2 = N - d_1 - x_3 - x_5$. Similar expressions may be obtained for $BR_3(A_2; N; q)$ and $BR_3(A_3; N; q)$.

We now turn our attention to the case in which the separate removals of A_i and A_j result in a Borda reversal in both of the reduced sets of two alternatives. Before proceeding to develope an expression for $BR_3(A_i, A_j; N; q)$ we demonstrate that if either A_i or A_j lies in second - place in the original Borda social ordering then the separate removals of A_i and A_j can not produce a Borda reversal in both reduced sets.

Suppose $B_1 > B_2$ and $B_1 > B_3$. That is

$$x_1 + 2x_2 + x_5 > x_3 + 2x_4 + x_6 \tag{9.10}$$

and

$$2x_1 + x_2 + x_3 > x_4 + x_5 + 2x_6 \qquad (9.11)$$

For a Borda reversal to occur following the removal of A_3, it is required that

$$x_3 + x_4 + x_6 > x_1 + x_2 + x_5 \qquad (9.12)$$

Likewise, a Borda reversal occurs when A_2 is removed if

$$x_4 + x_5 + x_6 > x_1 + x_2 + x_3 \qquad (9.13)$$

Now, (9.10) and (9.11) imply that

$$x_1 + x_2 > x_4 + x_6 \qquad (9.14)$$

whereas (9.12) and (9.13) imply that

$$x_4 + x_6 > x_1 + x_2 \qquad (9.15)$$

That is, inequalities (9.10), (9.11), (9.12) and (9.13) generate a contradiction and cannot therefore be realised simultaneously. Thus, the separate removals of A_2 and A_3 cannot produce a Borda reversal in both reduced sets. A similar result is obtained when originally $B_2 > B_1$ and $B_3 > B_1$, and once again A_2 and A_3 are separately removed. Therefore, when A_1 is in first - place or in third - place in the original Borda social ordering the separate removals of A_2 and A_3 cannot produce a Borda reversal in both reduced sets. Now, when A_1 is originally in first - place or in third - place, either A_2 or A_3 must occupy second - place. By interchanging the labels for the alternatives we may state, for all pairs A_i and A_j, $i \neq j$, that if either A_i or A_j occupies second - place in the original Borda social ordering then the separate removals of A_i and A_j cannot produce a Borda reversal in both reduced sets. An immediate corollary is that

$$BR_3(A_1, A_2, A_3; N; q) = 0 \qquad (9.16)$$

Consider alternatives A_1 and A_3. From the preceding analysis it follows that the separate removals of A_1 and A_3 can only result in a Borda reversal in both reduced sets provided the original Borda social ordering is of the form $B_1 > B_2 > B_3$ or $B_3 > B_2 > B_1$. Let us deal with the case $B_1 > B_2 > B_3$ first of all.

We wish to obtain all combinations of values of x_i, $i = 1, 2, ..., 6$, which produce the results : $B_1 > B_2 > B_3$, $B_2 > B_1$ when B_3 is removed, and $B_3 > B_2$ when B_1 is removed. That is, it is required that

$$x_1 + 2x_2 + x_5 > x_3 + 2x_4 + x_6 \quad (9.17a)$$

$$x_1 + 2x_3 + x_4 > x_2 + 2x_5 + x_6 \quad (9.17b)$$

and

$$x_3 + x_4 + x_6 > x_1 + x_2 + x_5 \quad (9.18a)$$

$$x_2 + x_5 + x_6 > x_1 + x_3 + x_4 \quad (9.18b)$$

Proceeding in the same manner as in the derivation of (5.3), we establish limits for B_1, B_2, a_1, a_2, and x_1, thereby determining the relevant values of x_i, $i = 1, 2, ..., 6$.

Since $B_1 + B_2 + B_3 = 3N$ the lowest value B_1 can assume is $N + 1$. From (9.18a) and (9.18b) it is apparent that $x_1 + x_2 + x_3 + x_5 \leq 2k$. That is, $a_1 + b_1 \leq 2k$. Also, $a_1 \leq k$. Hence, $B_1 = 2a_1 + b_1$ has an upper limit of $3k$. Now, $B_2 \geq B_3 + 1$, or $B_2 \geq 3N - B_1 - B_2 + 1$. Thus, $B_2 \geq \{(3N - B_1 + 2)/2\}$. An additional component in B_2's lower limit exists. From (9.18a) we have $x_3 + x_4 + x_6 \geq N - k$, or $x_3 + x_4 + N - a_1 - b_1 - x_4 \geq N - k$, or $x_3 \geq B_1 - a_1 - k$. When $x_3 = B_1 - a_1 - k$ then $x_4 = 0$, and $a_2 = x_3 + x_4$ reaches its lowest value for given B_1 and a_1. Now, for given B_1, B_2 reaches its lower limit when both a_2 and b_2 take

their lowest values. The lowest value of a_2 occurs when $x_3 = B_1 - a_1 - k$ and $a_1 = k$, in which event $a_2 = B_1 - 2k$. The lowest value of b_2 is reached when $N - a_2 - b_2$ is at its highest value, i.e. when $c_2 = k$ (9.18a). In this event, $b_2 = N - a_2 - c_2 = N - B_1 + k$. Hence, $B_2 \geq 2(B_1 - 2k) + N - B_1 + k$, or $B_2 \geq B_1 + N - 3k$. The lower limit of B_2 is therefore equal to $\max\left[\left\{(3N - B_1 + 2)/2\right\}, B_1 + N - 3k\right]$. Now, $B_2 \leq B_1 - 1$, but there is an additional component in B_2's upper limit. From (9.18b) we have $x_2 + x_5 + x_6 \geq N - k$, or $x_2 + N - a_2 - b_2 - x_2 + x_6 \geq N - k$, or $x_6 \geq B_2 - a_2 - k$. When $x_6 = B_2 - a_2 - k$ then $x_5 = 0$, and $a_3 = x_5 + x_6$ reaches its lowest value for given B_2 and a_2. Now, for given B_1, B_2 reaches its upper limit when B_3 takes its lowest value. In turn, B_3 reaches its lower limit when both a_3 and b_3 take their lowest values. The lowest value of a_3 occurs when $x_6 = B_2 - a_2 - k$ and $a_2 = k$, in which event $a_3 = B_2 - 2k$. The lowest value of b_3 is reached when $N - a_3 - b_3$ is at its highest value, i.e. when $c_3 = k$ (9.18b). In this event, $b_3 = N - a_3 - c_3 = N - B_2 + k$. Hence, $B_3 \geq 2(B_2 - 2k) + N - B_2 + k$, or $B_3 \geq B_2 + N - 3k$. Since $B_3 = 3N - B_1 - B_2$, we have $3N - B_1 - B_2 \geq B_2 + N - 3k$, or $B_2 \leq \left\{(2N + 3k - B_1)/2\right\}$. The upper limit of B_2 is therefore equal to
$\min\left[B_1 - 1, \left\{(2N + 3k - B_1)/2\right\}\right]$.

For given B_1 and B_2, a_1 reaches its lower limit when a_2 and a_3 attain their upper limits. When $B_2 > N$ we have from (9.18b) $a_2 \leq k$. Also, it is readily apparent that a_3 attains its upper limit when c_3 is at its maximum, i.e. when $c_3 = k$ (9.18b). Both upper limits may be reached simultaneously when $x_1 = x_4 = 0$. Thus, we may write $a_2 + c_3 \leq 2k$, or $a_2 + N - B_3 + a_3 \leq 2k$, or $N - a_1 + N + B_1 + B_2 - 3N \leq 2k$, or $a_1 \geq B_1 + B_2 - N - 2k$. However, when $B_2 < N$ the upper limit of a_2 is less than k, and the lower limit of a_1 must take this into account. From (9.18a) we have $a_2 + b_2 \geq N - k$, or $a_2 + B_2 - 2a_2 \geq N - k$, or $a_2 \leq B_2 + k - N$. When $a_2 = B_2 + k - N$ then $c_2 = k$. Therefore, $x_1 = 0$ and $c_3 = a_2 = x_3$. Thus, we may write

$a_2 + c_3 \leq 2(B_2 + k - N)$, or

$a_2 + N - B_3 + a_3 \leq 2B_2 + 2k - 2N$, or

$B_1 + B_2 - a_1 - N \leq 2B_2 + 2k - 2N$, or $a_1 \geq B_1 - B_2 + N - 2k$.

The lower limit of a_1 is therefore equal to $\max\left[B_1 + B_2 - N - 2k,\ B_1 - B_2 + N - 2k\right]$. The upper limit of a_1 also has two components. From (9.18a) we have $a_1 \leq k$. Further, for given B_1 and B_2, a_1 attains its upper limit when a_2 and a_3 take their lowest values. From (9.18b) it is apparent that $x_2 + x_5 + x_6 \geq N - k$, or $x_2 + N - a_2 - b_2 - x_2 + x_6 \geq N - k$, or $x_6 \geq a_2 + b_2 - k$, or $x_6 \geq B_2 - a_2 - k$. Now, when $x_6 = B_2 - a_2 - k$ then $x_5 = 0$, which implies that $a_3 \geq B_2 - a_2 - k$, or $a_2 + a_3 \geq B_2 - k$. Hence, $a_1 \leq N + k - B_2$. The upper limit of a_1 is therefore equal to $\min\left[k,\ N + k - B_2\right]$.

For given B_1, B_2, and a_1, the lower limit of a_2 is reached when c_2 takes its lowest value. From (9.18b) we have $c_1 + c_2 \geq N - k$, or $N - B_1 + a_1 + N - B_2 + a_2 \geq N - k$, or $a_2 \geq B_1 + B_2 - a_1 - N - k$. However, when $B_2 < N$ then a_1's upper limit equals k, whereupon from (9.18a) we have $x_3 + x_4 + x_6 \geq N - k$, or $x_3 + N - a_1 - b_1 - x_6 + x_6 \geq N - k$, or $x_3 \geq B_1 - a_1 - k$. When $x_3 = B_1 - a_1 - k$ then $x_4 = 0$ and $a_2 = B_1 - a_1 - k$. Therefore, the lower limit of a_2 equals

$\max\left[B_1 + B_2 - a_1 - N - k,\ B_1 - a_1 - k\right]$. The upper limit of a_2, it will be recalled, we previously established to be $\min\left[k,\ B_2 + k - N\right]$. Finally, the lower limit of x_1 takes the same form as in (5.3), while from (9.18a) and (9.18b) it is apparent that the upper limit of x_1 is equal to $\min\left[k - a_2,\ k - c_2\right]$.

We have established all combinations of values of x_i, $i = 1, 2, \ldots, 6$, which following the separate removals of A_1 and A_3 give rise to a Borda reversal in both reduced sets, provided the original Borda social ordering is $B_1 > B_2 > B_3$. To obtain the corresponding values of x_i, $i = 1, 2, \ldots, 6$, when the original social ordering is $B_3 > B_2 > B_1$ we simply interchange the labels for alternatives A_1 and A_3. Thus, corresponding to $x_1, x_2, x_3, x_4, x_5, x_6$ in case $B_1 > B_2 > B_3$ we have respectively $x_6, x_5, x_4, x_3, x_2, x_1$ in case $B_3 > B_2 > B_1$.

Employing the above findings in conjunction with the multinomial social choice model we obtain the following expression for the probability that the separate removals of A_1 and A_3 result in a Borda reversal in both reduced sets:

$$BR_3(A_1, A_3; N; q)$$

$$= \sum^{5} \frac{N!}{x_1!x_2!x_3!x_4!x_5!x_6!}\left[q_1^{x_1} q_2^{x_2} q_3^{x_3} q_4^{x_4} q_5^{x_5} q_6^{x_6} + q_1^{x_6} q_2^{x_5} q_3^{x_4} q_4^{x_3} q_5^{x_2} q_6^{x_1}\right] \quad (9.19)$$

where $\sum^{5}$ is a 5 - fold summation sign whose variables of summation have the following limits

$$N+1 \leq B_1 \leq 3k$$

$$\max\left[\{(3N-B_1+2)/2\}, B_1+N-3k\right] \leq B_2 \leq \min\left[B_1-1, \{(2N+3k-B_1)/2\}\right]$$

$$\max\left[B_1+B_2-N-2k, B_1-B_2+N-2k\right] \leq a_1 \leq \min\left[k, N+k-B_2\right] \qquad (9.20)$$

$$\max\left[B_1+B_2-a_1-N-k, B_1-a_1-k\right] \leq a_2 \leq \min\left[k, B_2+k-N\right]$$

$$\max\left[a_1-c_2, a_1-b_3, 0\right] \leq x_1 \leq \min\left[k-a_2, k-c_2\right]$$

Similar expressions may be obtained for $BR_3(A_1,A_2;\ N;\ q)$ and $BR_3(A_2,A_3;\ N;\ q)$.

By (9.1) the probability of a Borda reversal in case $m = 3$ is given by

$$BR_3(N;\ q) = \sum_{i=1}^{3} BR_3(A_i;\ N;\ q) - \sum_{i<j} BR_3(A_i,A_j;\ N;\ q)$$

$$+ BR_3(A_1,A_2,A_3;\ N;\ q) \qquad (9.21)$$

Substituting (9.8), (9.16), and (9.19) in (9.21) we obtain finally

$$BR_3(N;\, q) = \overset{3}{\Sigma} \frac{N!}{d_1!x_3!x_5!d_2!}\Big[(q_1+q_4)^{d_1}\, q_3^{x_3}\, q_5^{x_5}\, (q_2+q_6)^{d_2}$$

$$+ (q_1+q_4)^{d_2}\, q_3^{x_5}\, q_5^{x_3}\, (q_2+q_6)^{d_1}$$

$$+ (q_2+q_3)^{d_1}\, q_1^{x_3}\, q_6^{x_5}\, (q_4+q_5)^{d_2}$$

$$+ (q_2+q_3)^{d_2}\, q_1^{x_5}\, q_6^{x_3}\, (q_4+q_5)^{d_1}$$

$$+ (q_1+q_5)^{d_1}\, q_2^{x_3}\, q_4^{x_5}\, (q_3+q_6)^{d_2}$$

$$+ (q_1+q_5)^{d_2}\, q_2^{x_5}\, q_4^{x_3}\, (q_3+q_6)^{d_1}\Big]$$

$$+ \overset{5}{\Sigma} \frac{N!}{x_1!x_2!x_3!x_4!x_5!x_6!}\Big[q_1^{x_1}\, q_2^{x_2}\, q_3^{x_3}\, q_4^{x_4}\, q_5^{x_5}\, q_6^{x_6}$$

$$+ q_1^{x_6}\, q_2^{x_5}\, q_3^{x_4}\, q_4^{x_3}\, q_5^{x_2}\, q_6^{x_1}$$

$$+ q_1^{x_2}\, q_2^{x_1}\, q_3^{x_5}\, q_4^{x_6}\, q_5^{x_3}\, q_6^{x_4}$$

$$+ q_1^{x_4}\, q_2^{x_3}\, q_3^{x_6}\, q_4^{x_5}\, q_5^{x_1}\, q_6^{x_2}$$

$$+ q_1^{x_3}\, q_2^{x_4}\, q_3^{x_1}\, q_4^{x_2}\, q_5^{x_6}\, q_6^{x_5}$$

$$+ q_1^{x_5}\, q_2^{x_6}\, q_3^{x_2}\, q_4^{x_1}\, q_5^{x_4}\, q_6^{x_3}\Big] \quad (9.22)$$

where $\overset{3}{\Sigma}$ and $\overset{5}{\Sigma}$ are as defined in (9.9) and (9.20) respectively. Once again, as in previous expressions of this form the constituent summations have, where possible, been brought together under a single summation sign.

9.2 EFFECT OF GROUP SIZE AND CULTURE ON THE LIKELIHOOD OF A BORDA REVERSAL

By means of formula (9.22) the likelihood of a Borda reversal in case m = 3 was calculated in each of six cultures for group sizes N = 3, 4, ..., 100. Table 9.1 contains the resultant probability values. Culture (i) is the equiprobable culture; cultures (ii), (iii), and (vi) have the property of a single - peakedness; culture (iv) is the culture in which the likelihood of the paradox of voting is at its maximum; and culture (v) is a culture in which the plurality and Condorcet outcomes always agree. In respect of these cultures, $\sum_{i<j} BR_3(A_i, A_j; N; q)$ equals zero in all except culture (i) and culture (iv).

Davidson and Odeh (1972) demonstrated in case m = 3 that a Borda reversal following the removal of the original Borda winner cannot occur when N = 3, 4, and 6. The present findings extend this result by indicating that no Borda reversal of any type can occur when N = 3, 4, and 6.

When m = 3, Borda distortion is a special case of Borda reversal. Hence, the likelihood of occurrence of Borda distortion in a given culture is always less than or equal to that of a Borda reversal. A comparison of Tables 8.2 and 9.1 reveals that in culture (vi) when N > 100 Borda distortion is the only type of Borda reversal with any probability of occurrence.

In all cultures the probability of a Borda reversal is generally greater in an odd - sized group than in an even - sized group of similar magnitude. In the equiprobable culture, as the study by Fishburn (1974b) suggested, the likelihood of a Borda reversal increases steadily with N, its gradual rise unabated by N = 100. Depending on the culture, the probability of a Borda reversal may approach a limiting value of zero, e.g. culture (iii), or unity, e.g. culture (iv). Cultures (i) and (ii) would appear to have intermediate limiting values.

From a comparison of the odd - sized groups in Tables 6.1 and 9.1 it is apparent that in cultures where there is a finite probability of a cyclical majority, the phenomenon of a Borda reversal either has a similar or a greater probability of occurrence. For example, in the equiprobable culture when $N > 3$ the paradox of voting and the Borda reversed - order paradox are initially alike in terms of their probability of occurrence. However, by N = 21 the likelihood of a Borda reversal is twice as great as that of a cyclical majority, and this factor continues to increase with N.

Moreover, in cultures where the paradox of voting cannot arise, i.e. value - restricted cultures such as culture (vi), the likelihood of a Borda reversal can be considerable. Thus, to the extent that the occurrence of a Borda reversal is regarded as a deficiency of similar gravity to the occurrence of a cyclical majority, the Borda procedure would appear to compare unfavourably with the Condorcet procedure. However, it must be remembered that we are looking at the incidence of Borda reversals anywhere in the original social preference ordering. If reversals involving original Borda losers only are excluded then the probabilities in Table 9.1 will be reduced somewhat. Nevertheless, consideration of the values in Table 8.2 for the likelihood of Borda distortion, which is a special case of the type of Borda reversal which always involves the original Borda winner, indicates that this type of Borda reversal can still have a likelihood of occurrence which exceeds that of the paradox of voting. Thus, the reversed - order paradox is as serious a problem for the Borda procedure as the paradox of voting is for the Condorcet procedure.

TABLE 9.1

PROBABILITY OF A BORDA REVERSAL FOR N GROUP MEMBERS AND

m = 3 ALTERNATIVES IN EACH OF SIX CULTURES :

(i) q = (1/6, 1/6, 1/6, 1/6, 1/6, 1/6),

(ii) q = (1/4, 0, 1/4, 1/4, 0, 1/4),

(iii) q = (1/2, 0, 1/4, 0, 0, 1/4), ... continued overleaf

N	Culture (i)	Culture (ii)	Culture (iii)
3	.00000	.00000	.00000
4	.00000	.00000	.00000
5	.03858	.03906	.01953
6	.00000	.00000	.00000
7	.07352	.06409	.03418
8	.00440	.00342	.00171
9	.09985	.07594	.04037
10	.01152	.00801	.00441
11	.11943	.08028	.04025
12	.01971	.01227	.00708
13	.13433	.08105	.03664
14	.02802	.01576	.00908
15	.14599	.08038	.03166
16	.03604	.01852	.01016
17	.15535	.07933	.02656
18	.04357	.02076	.01039
19	.16305	.07836	.02193
20	.05058	.02266	.00999
49	.20969	.08172	.00101
50	.11272	.04163	.00083
99	.22956	.08834	.00001
100	.15269	.05657	.00001

TABLE 9.1 (continued)

(iv) q = (1/3, 0, 0, 1/3, 1/3, 0),

(v) q = (0, 5/12, 7/12, 0, 0, 0),

(vi) q = (.1, .2, .6, 0, .1, 0).

N	Culture (iv)	Culture (v)	Culture (vi)
3	.00000	.00000	.00000
4	.00000	.00000	.00000
5	.61728	.59076	.11540
6	.00000	.00000	.00000
7	.57613	.50256	.18768
8	.56333	.41321	.03871
9	.51852	.43974	.21699
10	.52279	.37663	.08651
11	.73881	.70381	.23583
12	.47838	.34524	.12438
13	.73943	.64946	.25408
14	.70320	.56599	.15402
15	.70618	.60194	.26974
16	.69985	.53554	.17977
17	.82317	.76443	.28237
18	.67377	.50661	.20253
19	.83141	.72506	.29339
20	.79549	.65281	.22232
49	.97750	.87217	.39601
50	.97014	.83845	.37534
99	.99078	.94417	.45832
100	.99876	.93300	.45325

CHAPTER 10

SOCIAL DECISION SCHEME MODELS

10.1 SIMPLE MODELS

In this chapter we turn from the investigation of formal methods of collective decision making to study the implicit, largely unverbalised social choice schemes employed by informal groups. Thus, the emphasis of the present chapter contrasts with that of preceding ones in so far as descriptive rather than normative considerations predominate. Otherwise, the basic paradigm, a group of N members selecting one of m alternatives A_1, A_2,...., A_m, remains the same.

Davis (1973) first proposed that the social interaction involved in group decision making might be regarded as the operation of a social decision scheme. He suggested that opinions expressed in group debate may be viewed as votes which are transformed by a group "combinatorial mechanism" into a collective decision. Now, the discovery of the implicit social decision schemes employed by various types of group is a matter for empirical research. Nevertheless, in so far as models of the group combinatorial process are available then impetus and direction are imparted to empirical research and unexpected effects of social systems may be predicted and investigated. However, in this respect, Davis (1973) comments that "one of the great misfortunes of combinatorial algebra, upon which the general social decision scheme model is based, is that there rarely exist explicit expressions which can be studied directly for their implications " (p.122).

It is the aim of the present chapter to address itself to this problem. Explicit expressions are developed for the probability that a group chooses a particular alternative, in each of a number of special cases of the general social decision scheme model. The models are compared in terms of the predictions yielded by their probability expressions.

In the general social decision scheme model enunciated by Davis (1973), only the first preferences of the group members are taken into account. However, the model may readily be extended to encompass each member's full preference ordering of the alternatives. With m alternatives there are m! linear preference orderings and consequently $\binom{N + m! - 1}{N}$ distinguishable distributions of the members over the preference orderings. Let π_i denote the probability of the i^{th} such distribution or preference configuration, where $\Sigma \pi_i = 1$. Also, let d_{ij} denote the probability that, given the i^{th} preference configuration, the group decides on alternative A_j. Now, $\sum_j d_{ij} \leq 1$. The likelihood that the group will fail to reach a clear - cut decision in the case of the i^{th} preference configuration is given by $1 - \sum_j d_{ij}$. The d_{ij} "may be regarded as parameters reflecting tradition, norms, task features, interpersonal tendencies, local conditions, or some combination of these within which the group is embedded" (Davis, 1973). From elementary probability considerations, we have

$$p(i^{th} \text{ configuration} \cap \text{group chooses } A_j)$$
$$= p(i^{th} \text{ configuration}).\ p(\text{group chooses } A_j \mid i^{th} \text{ configuration})$$
$$= \pi_i\, d_{ij} \quad . \qquad (10.1)$$

If $p(A_h)$ is the probability that the group selects alternative A_h then the general model states that

$$p(A_h) = \sum_i \pi_i\, d_{ih} \ . \qquad (10.2)$$

In order to model the process by which the π_i are generated, Davis employed assumptions which are identical to those outlined in chapter 2. Hence, the probability π_i that the i^{th} preference configuration occurs is provided by the multinomial choice model (2.2) .

Since it would be unreasonable to expect the d_{ij}, to remain invariant across groups, tasks, and situations, Davis and subsequent researchers (e.g. Kerr et al, 1975; Laughlin et al, 1975) have considered several special cases of the general model. All of these special cases, being derived from Davis's presentation of the general model, deal only with the first preferences of the group members, and almost all lack explicit expressions for $p(A_h)$.

Four such special cases of the general model are examined in the present section:

(i) majority, in which the group selects an alternative if at least a majority advocates it; that is, given the i^{th} preference configuration,

$$d_{ih} = 1 \quad \text{if } a_h > N/2,$$
$$= 0 \quad \text{otherwise ;} \qquad (10.3)$$

(ii) proportionality, in which the probability of a particular group decision is the proportion or members advocating that alternative; that is, given the i^{th} preference configuration,

$$d_{ih} = a_h/N; \qquad (10.4)$$

(iii) equiprobability, in which the group choice is equally likely to be any of those alternatives advocated by one or more members; that is, given the i^{th} preference configuration,

$$d_{ih} = 1/(m - z), \qquad (10.5)$$

where z is the number of alternatives with exactly zero first preference votes;

(iv) highest expected value, in which the group choice is the alternative, of those advocated by at least one member, with the highest expected value; that is, given the i^{th} preference configuration,

$$d_{ih} = 1 \quad \text{if } EV(A_h) > EV(A_j), \text{ for all } j \neq h,\ a_j > 0, \text{ and } a_h > 0,$$
$$= 0 \quad \text{otherwise}, \qquad (10.6)$$

where $EV(A_h)$ is the expected value of A_h, h = 1, 2, ..., m, and provides a rank ordering of the alternatives in terms of a dimension relevant to group performance.

Before proceeding to develope an expression for $p(A_h)$ in each of these four social decision scheme models, we note that the plurality, Condorcet, and Borda procedures, studied in previous chapters, may also be regarded as special cases of the general model. Indeed, the plurality social decision scheme model has received extensive investigation (e.g. Davis et al, 1974; Kerr et al, 1975). Thus, the results of earlier chapters are also relevant to the present discussion.

In terms of the theory of social decision schemes the plurality model, given the i^{th} preference configuration, is defined

$$d_{ih} = 1 \quad \text{if } a_h > a_j, \text{ for all } j \neq h; \qquad (10.7)$$
$$= 0 \quad \text{otherwise.}$$

For given N, m, and p, the probability that A_h is the plurality winner is

$$p(A_h) = P_m(A_h; N; p) \qquad (10.8)$$

where $P_m(A_h; N; p)$ may be evaluated by means of expression (3.1).

The Condorcet model, given the i^{th} preference configuration, is defined

$$d_{ih} = 1 \quad \text{if } q_{hj} > 1/2, \text{ for all } j \neq h; \qquad (10.9)$$
$$= 0 \quad \text{otherwise.}$$

For given N, m, and q, the probability that A_h is the Condorcet winner is

$$p(A_h) = C_m(A_h; N; q) \qquad (10.10)$$

where $C_m(A_h; N; q)$ may be evaluated by means of one of the expressions in chapter 4.

The Borda model, given the i^{th} preference configuration, is defined

$$d_{ih} = 1 \quad \text{if } B_h > B_j, \text{ for all } j \neq h; \qquad (10.11)$$
$$= 0 \quad \text{otherwise.}$$

For given N, m, and q, the probability that A_h is the Borda winner is

$$p(A_h) = B_m(A_h; N; q) \qquad (10.12)$$

where $B_m(A_h; N; q)$ may be evaluated when m = 3 by means of expression (5.2)

Of the seven foregoing social decision scheme models, only the Condorcet and Borda models take into consideration each individual's full preference ordering. In the majority, plurality, Condorcet, and Borda models, a state of collective indecision is possible, since in each case $\sum_j d_{ij} \leq 1$. In the proportionality, equiprobability, and highest expected value models collective indecision cannot occur since in each case $\sum_j d_{ij} = 1$.

Expressions for $p(A_h)$ are now developed in the case of the majority, proportionality, equiprobability, and highest expected value social decision scheme models.

Majority. The collective decision is the alternative, if one exists, which gains more than half of the first preference votes.

The majority social decision scheme is a truncated version of the plurality scheme. If 2k + 2 = N when N is even and 2k + 1 = N when N is odd, then modifying the limits of summation given by (3.2) to take account of the fact that the winner receives at least N - k votes, we obtain the following expression for the probability that A_1 is the majority winner :

$$p(A_1) = \sum^{m-1} \frac{N!}{a_1!a_2!\cdots a_m!} p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m} \qquad (10.13)$$

where $\sum^{m-1}$ is an (m-1) - fold summation whose variables of summation have the following limits

$$
\begin{aligned}
N - k &\leq a_1 &&\leq N \\
0 &\leq a_2 &&\leq N - a_1 \\
0 &\leq a_3 &&\leq N - a_1 - a_2 \\
&\vdots \\
0 &\leq a_{m-1} &&\leq N - \sum_{j=1}^{m-2} a_j \,.
\end{aligned}
\tag{10.14}
$$

Application of the multinomial theorem (Feller, 1968) to variables $a_2, a_3, \ldots, a_m$ yields

$$
p(A_1) = \sum_{a_1=N-k}^{N} \binom{N}{a_1} p_1^{a_1} (1-p_1)^{N-a_1} \tag{10.15}
$$

Similar expressions may be obtained for $p(A_h)$, $h = 2, 3, \ldots, m$.

Expression (10.15) consists of the upper section of a binomial distribution. Hence, if $p_1 > 1/2$ then, by (3.6), $p(A_1)$ tends to unity as N becomes large.

On the other hand, given p_1 and N, if m is permitted to vary and/or the other p_j are free to take any values subject to the constraint $\sum_{j=2}^{m} p_j = 1 - p_1$, then $p(A_1)$ remains constant.

The majority and plurality schemes are equivalent in the following circumstances : (a) when $m = 2$; (b) when $N = 3$; and (c) when $m = 3$ and $N = 5$.

<u>Proportionality</u>. The probability that the collective decision is alternative A_h is equal to the proportion of first preference votes cast in favour of A_h.

Consider alternative A_1. According to the proportionality scheme, the probability that A_1 is the winner is

$$p(A_1) = \sum^{m-1} \frac{N!}{a_1!a_2!\ldots.a_m!} p_1^{a_1} p_2^{a_2} \ldots . p_m^{a_m} \cdot \frac{a_1}{N} \tag{10.16}$$

where $\sum^{m-1}$ is an (m-1) - fold summation whose variables of summation have the following limits

$$\begin{aligned} 0 &\leq a_1 \leq N \\ 0 &\leq a_2 \leq N - a_1 \\ 0 &\leq a_3 \leq N - a_1 - a_2 \\ &\vdots \\ 0 &\leq a_{m-1} \leq N - \sum_{j=1}^{m-2} a_j \, . \end{aligned} \tag{10.17}$$

Cancelling terms and setting $f = a_1 - 1$, we have

$$p(A_1) = p_1 \sum^{m-1} \frac{(N-1)!}{f!a_2!a_3!\ldots.a_m!} p_1^{f} p_2^{a_2} p_3^{a_3} \ldots . p_m^{a_m} \tag{10.18}$$

where $\sum^{m-1}$ is an (m-1) - fold summation whose variables of summation have the following limits

$$\begin{aligned} 0 &\leq f \leq N - 1 \\ 0 &\leq a_2 \leq N - 1 - f \\ 0 &\leq a_3 \leq N - 1 - f - a_2 \\ &\vdots \\ 0 &\leq a_{m-1} \leq N - 1 - f - \sum_{j=2}^{m-2} a_j \end{aligned} \tag{10.19}$$

Hence, by the multinomial theorem,

$$\begin{aligned} p(A_1) &= p_1(p_1 + p_2 + \ldots . + p_m)^{N-1} \\ &= p_1 \end{aligned} \tag{10.20}$$

In general, by similar reasoning,

$$p(A_h) = p_h \tag{10.21}$$

Clearly, given p_h, $p(A_h)$ is independent of : (a) N ; (b) m ; and (c) fluctuations in the other p_j subject to the constraint $\sum_{j \neq h} p_j = 1 - p_h$.

An alternative way to arrive at expression (10.20) is to note that expression (10.16) is simply the expectation of a multinomial random variable divided by N, i.e. $(Na_1)/N$.

Equiprobability. The collective decision is equally likely to be any of the alternatives which have at least one first preference vote.

Let G_{hi} denote the probability that i alternatives, one of which is A_h, share the N votes such that each of them receives at least one vote. Equivalently, G_{hi} represents the probability that exactly m - i alternatives, none of which is A_h, each receives zero votes. The equiprobability social decision scheme model states that the probability that A_h wins is

$$p(A_h) = \sum_{i=1}^{m} (1/i)\, G_{hi} \tag{10.22}$$

Let G^o_{hi} denote the probability that exactly m - i alternatives, one of which is A_h, each receives zero votes. Equivalently, G^o_{hi} represents the probability that i alternatives, none of which is A_h, share the N votes. If $G_{(m-i)}$ denotes the probability that exactly m - i out of m alternatives receive zero votes, we have

$$G_{(m-i)} = G_{hi} + G^o_{hi} \tag{10.23}$$

We proceed by deriving expressions for $G_{(m-i)}$ and G^{o}_{hi}, whereupon G_{hi} may be determined by means of (10.23).

Define

$$
\begin{aligned}
u_e &= p_e^N \\
u_{ef} &= (p_e + p_f)^N \\
u_{efg} &= (p_e + p_f + p_g)^N \\
&\vdots \\
u_{123....m} &= (p_1 + p_2 + p_3 + ... + p_m)^N = 1
\end{aligned}
$$

Thus, u_e represents the probability that alternative A_e receives all N votes, u_{ef} represents the probability that A_e and A_f share all N votes, u_{efg} represents the probability that A_e, A_f, and A_g share all N votes, and so on. Equivalently, u_e, u_{ef}, u_{efg}, ... represent respectively the probability that the other m - 1, m - 2, m - 3, ... alternatives receive zero votes. Also, define

$$
\begin{aligned}
S_{m-1} &= \sum_{e=1}^{m} u_e \\
S_{m-2} &= \sum_{e<f} u_{ef} \\
S_{m-3} &= \sum_{e<f<g} u_{efg} \\
&\vdots \\
S_0 &= u_{123....m} = 1
\end{aligned}
$$

The probability that exactly m - i out of m alternatives receive zero votes may be expressed (Feller, 1968, chapter IV) as

$$G_{(m-i)} = S_{m-i} - \binom{m-i+1}{m-i} S_{m-i+1} + \binom{m-i+2}{m-i} S_{m-i+2}$$

$$- \ldots + (-1)^i \binom{m}{m-i} S_m \qquad (10.24)$$

Since it is not possible for all alternatives simultaneously to obtain zero votes, it follows that $S_m = 0$.

An expression for G^o_{hi} may be obtained by following a similar line of reasoning to that employed in establishing (10.24). By definition, G^o_{hi} is the probability that A_h receives zero votes <u>and</u> exactly m - i - 1 out of the remaining m - 1 alternatives receive zero votes. Define

$$S^o_{m-2} = \sum_{e \neq f} u_e$$

$$S^o_{m-3} = \sum_{\substack{e<f \\ e,f \neq h}} u_{ef}$$

$$S^o_{m-4} = \sum_{\substack{e<f<g \\ e,f,g \neq h}} u_{efg}$$

$$\vdots$$

$$S^o_0 = u_{12\ldots(h-1)(h+1)\ldots m}$$

Each sum S^o_{m-k} , k = 2, 3, ..., m, consists of all u - terms, with k - 1 subscripts, which do not contain the term p_h. The probability that m - 1 - i out of m - 1 alternatives receive zero votes <u>and</u> A_h receives zero votes is therefore

$$G^o_{hi} = S^o_{m-i-1} - \binom{m-i}{m-i-1} S^o_{m-i} + \binom{m-i+1}{m-i-1} S^o_{m-i+1}$$

$$- \ldots + (-1)^i \binom{m-1}{m-i-1} S^o_{m-1} \qquad (10.25)$$

Since it is not possible for A_h to receive zero votes and all m - 1 remaining alternatives to receive zero votes, it follows that $S^o_{m-1} = 0$.

By (10.23) an expression for G_{hi} may be found by subtracting (10.25) from (10.24). Thus,

$$G_{hi} = \sum_{k=0}^{i-1} (-1)^k \left[\binom{m-i+k}{m-i} S_{m-i+k} - \binom{m-i+k-1}{m-i-1} S^o_{m-i+k-1} \right] \quad (10.26)$$

In order to simplify (10.26), we define

$$S^h_{m-1} = u_h$$

$$S^h_{m-2} = \sum_{e \neq h} u_{he}$$

$$S^h_{m-3} = \sum_{\substack{e<f \\ e,f \neq h}} u_{hef}$$

$$\vdots$$

$$S^h_0 = u_{123....m} = 1$$

Each sum S^h_{m-k}, k = 1, 2, ..., m, consists of all u - terms, with k subscripts, which contain the term p_h. Clearly, it follows from the above definitions that

$$S_{m-j} = S^h_{m-j} + S^o_{m-j-1} \quad (10.27)$$

Thus, we may write (10.26) as

$$G_{hi} = \sum_{k=0}^{i-1} (-1)^k \left[\binom{m-i+k}{m-i} (S^h_{m-i+k} + S^o_{m-i+k-1}) - \binom{m-i+k-1}{m-i-1} S_{m-i+k-1} \right] \quad (10.28)$$

Employing the relationship (Feller, 1968)

Since it is not possible for A_h to receive zero votes and all $m - 1$ remaining alternatives to receive zero votes, it follows that $S^o_{m-1} = 0$.

By (10.23) an expression for G_{hi} may be found by subtracting (10.25) from (10.24). Thus,

$$G_{hi} = \sum_{k=0}^{i-1} (-1)^k \left[\binom{m-i+k}{m-i} S_{m-i+k} - \binom{m-i+k-1}{m-i-1} S^o_{m-i+k-1} \right] \qquad (10.26)$$

In order to simplify (10.26), we define

$$S^h_{m-1} = u_h$$

$$S^h_{m-2} = \sum_{e \neq h} u_{he}$$

$$S^h_{m-3} = \sum_{\substack{e<f \\ e,f \neq h}} u_{hef}$$

$$\vdots$$

$$S^h_0 = u_{123....m} = 1$$

Each sum S^h_{m-k}, $k = 1, 2, ..., m$, consists of all u - terms, with k subscripts, which contain the term p_h. Clearly, it follows from the above definitions that

$$S_{m-j} = S^h_{m-j} + S^o_{m-j-1} \qquad (10.27)$$

Thus, we may write (10.26) as

$$G_{hi} = \sum_{k=0}^{i-1} (-1)^k \left[\binom{m-i+k}{m-i} (S^h_{m-i+k} + S^o_{m-i+k-1}) - \binom{m-i+k-1}{m-i-1} S_{m-i+k-1} \right] \qquad (10.28)$$

Employing the relationship (Feller, 1968)

$$\binom{a}{b} = \binom{a-1}{b} + \binom{a-1}{b-1}$$

we obtain

$$G_{hi} = \sum_{k=0}^{i-1} (-1)^k \left[\binom{m-i+k}{m-i} S^h_{m-i+k} + \binom{m-i+k-1}{m-i} S^o_{m-i+k-1} \right] \quad (10.29)$$

Pairing S^h_{m-i+k} with S^o_{m-i+k} allows (10.29) to be written as

$$G_{hi} = \sum_{k=0}^{i-1} (-1)^k \binom{m-i+k}{m-i} \left[S^h_{m-i+k} - S^o_{m-i+k} \right] \quad (10.30)$$

The probability that A_h is the winner according to the equiprobability social decision scheme model is found by substituting (10.30) in (10.22). Thus,

$$p(A_h) = \sum_{i=1}^{m} \sum_{k=0}^{i-1} (1/i)(-1)^k \binom{m-i+k}{m-i} \left[S^h_{m-i+k} - S^o_{m-i+k} \right] \quad (10.31)$$

Setting j = i - k and rearranging the order of summation gives

$$p(A_h) = \sum_{j=1}^{m} \sum_{i=j}^{m} (1/i)(-1)^{i-j} \binom{m-j}{m-i} \left[S^h_{m-j} - S^o_{m-j} \right] \quad (10.32)$$

Finally, repeated application of the relationship

$$\binom{a}{b} = \binom{a-1}{b} + \binom{a-1}{b-1}$$

yields

$$p(A_h) = \sum_{j=1}^{m} \frac{(j-1)!(m-j)!}{m!} \left[S^h_{m-j} - S^o_{m-j} \right] \quad (10.33)$$

where $S^o_{m-1} = 0$.

For example, if m = 4 and h = 1, we obtain

$$p(A_1) = \frac{1}{4}\ p_1^N$$

$$+ \frac{1}{12}\left[(p_1 + p_2)^N + (p_1 + p_3)^N + (p_1 + p_4)^N - p_2^N - p_3^N - p_4^N\right]$$

$$+ \frac{1}{12}\left[(p_1 + p_2 + p_3)^N + (p_1 + p_2 + p_4)^N + (p_1 + p_3 + p_4)^N - (p_2 + p_3)^N - (p_2 + p_4)^N - (p_3 + p_4)^N\right]$$

$$+ \frac{1}{4}\left[1 - (p_2 + p_3 + p_4)^N\right] \qquad (10.34)$$

Since $S_0^h = 1$, expression (10.33) consists essentially of the term 1/m together with a number of terms of magnitude less than unity each raised to the power N. Therefore, as N becomes large, $p(A_h)$ converges on the limiting value 1/m. Indeed, the limiting value is approached fairly quickly. For example, when m = 3 and $p_1 = .7$, $p_2 = .2$, and $p_3 = .1$, we find that $p(A_h)$, h = 1, 2, 3, displays the following behaviour as N increases :

N	$p(A_1)$	$p(A_2)$	$p(A_3)$
5	.54	.30	.16
10	.42	.35	.23
25	.35	.34	.31
50	.33	.33	.33

Clearly, by N = 25 the $p(A_h)$, h = 1, 2, 3, are quite close to their limiting value of 1/3, and by N = 50 convergence is complete to two decimal places. Note how, around N = 10, $p(A_2)$ overshoots its eventual limiting value. This happens because p_3 is small and, consequently, the behaviour of $p(A_2)$ initially resembles that which occurs in case m = 2 when the limiting value is 1/2.

Highest expected value. When m alternatives may be rank ordered in terms of their expected values on a dimension relevant to group performance, the collective decision is the alternative with the highest expected value, provided the alternative has at least one vote. Otherwise, the collective decision is the alternative with the next highest expected value given that it has at least one vote, and so on.

Let A_1 be the alternative with the highest expected value, A_2 the alternative with the second highest expected value,......, and A_m the alternative with the lowest expected value. It follows that

$$p(A_1) = 1 - p(a_1 = 0)$$
$$= 1 - (p_2 + p_3 + \ldots\ p_m)^N \qquad (10.35)$$

Likewise,

$$p(A_2) = p(a_1 = 0) - p(a_1 = 0 \cap a_2 = 0)$$
$$= (p_2 + p_3 + \ldots + p_m)^N - (p_3 + p_4 + \ldots + p_m)^N \qquad (10.36)$$

and

$$p(A_3) = p(a_1 = 0 \cap a_2 = 0) - p(a_1 = 0 \cap a_2 = 0 \cap a_3 = 0)$$
$$= (p_3 + p_4 + \ldots + p_m)^N - (p_4 + p_5 + \ldots + p_m)^N \qquad (10.37)$$

In general, we have

$$p(A_h) = \left(\sum_{j=h}^{m} p_j\right)^N - \left(\sum_{j=h+1}^{m} p_j\right)^N \quad \text{if } h < m; \qquad (10.38)$$
$$= a_m^N \quad \text{if } h = m.$$

Thus, the highest expected value model predicts that if $EV(A_h) > EV(A_j)$, all $j \neq h$, then $p(A_h)$ approaches unity as N becomes large provided that $p_h > 0$. In practice, the asymptote is approached fairly rapidly. For example, if $p_h = .1$ then

$p(A_h)$ increases with N as follows :

N	$p(A_h)$
10	.65
20	.88
40	.99

Another implication of this model is that, given p_h and N, if A_h has the highest or the lowest expected value, $p(A_h)$ remains constant irrespective of variations in m and/or the other p_j subject to the constraint $\sum_{j \neq h} p_j = 1 - p_h$.

It will be observed that expression (10.35) corresponds to the Lorge and Solomon (1955) Model A, describing group problem solving behaviour, in which the group adopts the correct solution if any group member suggests it.

10.2 COMPLEX MODELS

Three complex social decision scheme models are investigated in this section. Like the simple schemes previously considered, they are special cases of the general model. Each, in effect, represents the union of two simple models. The three models are: (i) majority if A_g, proportionality otherwise: in which the group selects A_g if at least a majority advocates it, otherwise the probability of a particular group choice is the proportion of members preferring that alternative; that is, given the i[th] preference configuration,

$$\begin{aligned} d_{ih} &= 1 \quad \text{if } h = g \text{ and } a_g > N/2; \\ &= a_h/N \quad \text{otherwise.} \end{aligned} \qquad (10.39)$$

(ii) majority with proportionality: in which the group selects an alternative if at least a majority advocates it, otherwise the probability of a particular group choice is the proportion of members preferring that alternative; that is, given the i^{th} preference configuration,

$$\begin{aligned} d_{ih} &= 1 \quad \text{if } a_h > N/2; \\ &= a_h/N \quad \text{otherwise.} \end{aligned} \tag{10.40}$$

(iii) majority with equiprobability: in which the group chooses an alternative if at least a majority prefers it, otherwise the group decision is equally likely to be any of those alternatives advocated by at least one member; that is, given the i^{th} preference configuration,

$$\begin{aligned} d_{ih} &= 1 \quad \text{if } a_h > N/2 ; \\ &= 1/(m - z) \text{ otherwise ,} \end{aligned} \tag{10.41}$$

where z is the number of alternatives with exactly zero votes.

All three complex models have the property $\sum_{h=1}^{m} d_{ih} = 1$. That is, none of the complex models considered permits a state of collective indecision. Expressions for $p(A_h)$ in each model are now derived.

Majority if A_g, proportionality otherwise. The collective decision is A_g if at least a majority advocates it; otherwise the probability that the collective decision is A_h equals the proportion of votes received by A_h, $h = 1, 2, \ldots , m$.

Consider alternative A_g. Let $2k + 1 = N$ if N is odd and $2k + 2 = N$ if N is even. Making use of the result (10.15) which yields the probability that a particular alternative is the majority winner, we may express the probability that A_g is the winner under the present scheme as

$$p(A_g) = p(a_g \geq N - k) + \sum_{a_g=0}^{N-k-1} p(a_g) \cdot \frac{a_g}{N}$$

$$= \sum_{a_g=N-k}^{N} \binom{N}{a_g} p_g^{a_g} (1 - p_g)^{N-a_g}$$

$$+ \sum_{a_g=0}^{N-k-1} \binom{N}{a_g} p_g^{a_g} (1 - p_g)^{N-a_g} \cdot \frac{a_g}{N} \qquad (10.42)$$

Let us concentrate for the moment on the second summation in (10.42). Cancelling terms and readjusting the limits of summation allows us to write the second summation as

$$\sum_{a_g=0}^{N-k-2} \binom{N-1}{a_g} p_g^{a_g+1} (1 - p_g)^{N-a_g-1}$$

which in turn equals

$$p_g \cdot \left(1 - \sum_{a_g=N-k-1}^{N-1} \binom{N-1}{a_g} p_g^{a_g} (1 - p_g)^{N-a_g-1} \right) .$$

Multiplying out and substituting back into (10.42) gives

$$p(A_g) = \sum_{a_g=N-k}^{N} \binom{N}{a_g} p_g^{a_g} (1 - p_g)^{N-a_g} + p_g$$

$$- \sum_{a_g=N-k-1}^{N-1} \binom{N-1}{a_g} p_g^{a_g+1} (1 - p_g)^{N-a_g-1}$$

$$= p_g + \sum_{a_g=N-k}^{N} \left[\binom{N}{a_g} - \binom{N-1}{a_g-1} \right] p_g^{a_g} (1 - p_g)^{N-a_g}$$

so that we have finally

$$p(A_g) = p_g + \sum_{a_g=N-k}^{N-1} \binom{N-1}{a_g} p_g^{a_g} (1 - p_g)^{N-a_g} \qquad (10.43)$$

Consider now alternative A_h, $h \neq g$. From (10.39) it is clear that in those preference configurations where alternative A_g receives more than N/2 votes we have $d_{ih} = 0$, $h \neq g$. Otherwise $d_{ih} = a_h/N$. Thus, A_h can win only when A_g does not command a majority of the votes. The probability that A_g does not receive a majority of the votes is given by

$$p(a_g < N - k) = \sum_{a_g=0}^{N-k-1} \binom{N}{a_g} p_g^{a_g} (1 - p_g)^{N-a_g} \qquad (10.44)$$

Expanding (10.44) by means of the binomial theorem (Feller, 1968) gives

$$p(a_g < N - k) = \sum_{a_g=0}^{N-k-1} \sum_{a_h=0}^{N-a_g} \frac{N!}{a_g!a_h!(N-a_g-a_h)}$$

$$\cdot\ p_g^{a_g} p_h^{a_h} (1 - p_g - p_h)^{N-a_g-a_h} \qquad (10.45)$$

To obtain $p(A_h)$ we multiply (10.45) by a_h/N. After cancelling terms and readjusting the limits of summation for a_h, we have

$$p(A_h) = \sum_{a_g=0}^{N-k-1} \sum_{a_h=0}^{N-a_g-1} \frac{(N-1)!}{a_g!a_h!(N-a_g-a_h-1)!}$$

$$\cdot\ p_g^{a_g} p_h^{a_h+1} (1-p_g-p_h)^{N-a_g-a_h-1} \qquad (10.46)$$

Expression (10.46) may be simplified by removing the factor p_h and applying the binomial theorem. Hence, we have finally

$$p(A_h) = p_h \sum_{a_g=0}^{N-k-1} \binom{N-1}{a_g} p_g^{a_g} (1 - p_g)^{N-a_g-1} \qquad (10.47)$$

If we define

$$W = \sum_{a_g=N-k}^{N-1} \binom{N-1}{a_g} p_g^{a_g} (1 - p_g)^{N-a_g-1}$$

then alternative versions of (10.43) and (10.47) are

$$p(A_g) = p_g + (1 - p_g)W \qquad (10.48)$$

and

$$p(A_h) = p_h . (1 - W) \quad \text{all } h \neq g \qquad (10.49)$$

Since W represents the upper section of a binomial distribution, by (3.6) it follows that if $p_g > 1/2$ then W approaches unity as N becomes large ; and if $p_g < 1/2$ then W approaches zero as N becomes large. Hence, from (10.48) and (10.49) it is evident that as N becomes large: $p(A_g)$ has a limiting value which equals unity if $p_g > 1/2$, and p_g if $p_g < 1/2$; and $p(A_h)$ has a limiting value which equals zero if $p_g > 1/2$, and p_h if $p_g < 1/2$.

Given p_g and N, $p(A_g)$ remains constant irrespective of variations in m and/or the other p_h, $h \neq g$, subject to the constraint $\sum_{h \neq g} p_h = 1 - p_g$. Given p_h, p_g, and N, $p(A_h)$ remains constant irrespective of fluctuations in m and/or the other p_j, $j \neq h$, $j \neq g$, subject to the constraint $\sum_{j \neq g,h} p_j = 1 - p_g - p_h$.

<u>Majority with proportionality</u>. The collective decision is the alternative, if one exists, which possesses more than half of the first preference votes; if such an alternative does not exist then the likelihood that the collective decision is

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Given p_g and N, $p(A_g)$ remains constant irrespective of variations in m and/or the other p_h, $h \neq g$, subject to the constraint $\sum_{h\neq g} p_h = 1 - p_g$. Given p_h, p_g, and N, $p(A_h)$ remains constant irrespective of fluctuations in m and/or the other p_j, $j \neq h$, $j \neq g$, subject to the constraint $\sum_{j\neq g,h} p_j = 1 - p_g - p_h$.

<u>Majority with proportionality</u>. The collective decision is the alternative, if one exists, which possesses more than half of the first preference votes; if such an alternative does not exist then the likelihood that the collective decision is

alternative A_h equals the proportion of votes cast for A_h, h = 1, 2, ..., m.

Let 2k + 2 = N if N is even and 2k + 1 = N if N is odd. The probability that A_h is the winner under the majority with proportionality scheme may be written as

$$p(A_h) = p(a_h \geq N - k) + p(A_h \text{ when } a_h < N - k) \qquad (10.50)$$

Since (10.15) affords an expression for $p(a_h \geq N - k)$, we concentrate our attention on the second term in (10.50), which may be expressed as

$$p(A_h \text{ when } a_h < N - k)$$

$$= \sum_{a_h=0}^{N-k-1} p(a_h \cap (a_j < N - k, \text{ all } j \neq h)) \cdot \frac{a_h}{N} \qquad (10.51)$$

Ignoring the factor a_h/N for the moment, it will be observed that the summation represents the probability that no majority winner emerges. Employing once again (10.15) for the likelihood that a given alternative is the majority winner, we are able to express the probability that no majority winner emerges as follows:

$$p(\text{no majority winner})$$

$$= 1 - \sum_{j=1}^{m} p(A_j)$$

$$= 1 - \sum_{j=1}^{m} \sum_{a_j=N-k}^{N} \binom{N}{a_j} p_j^{a_j} (1 - p_j)^{N-a_j}$$

$$= 1 - \sum_{a_h=N-k}^{N} \binom{N}{a_h} p_h^{a_h} (1 - p_h)^{N-a_h}$$

$$- \sum_{j \neq h} \sum_{a_j=N-k}^{N} \binom{N}{a_j} p_j^{a_j} (1 - p_j)^{N-a_j}$$

$$= \sum_{a_h=0}^{N-k-1} \binom{N}{a_h} p_h^{a_h} (1 - p_h)^{N-a_h}$$

$$- \sum_{j \neq h} \sum_{a_j=N-k}^{N} \binom{N}{a_j} p_j^{N-a_j} (1 - p_j)^{a_j} \qquad (10.52)$$

Expanding the second expression in (10.52) by means of the binomial theorem, and multiplying both sides of (10.52) by a_h/N, we obtain

$p(A_h \text{ when } a_h < N - k)$

$$= \sum_{a_h=0}^{N-k-1} \binom{N}{a_h} p_h^{a_h} (1 - p_h)^{N-a_h} \cdot \frac{a_h}{N}$$

$$- \sum_{j \neq h} \sum_{a_j=N-k}^{N} \sum_{a_h=0}^{N-a_j} \frac{N!}{a_j! a_h! (N-a_j-a_h)!}$$

$$\cdot p_j^{a_j} p_h^{a_h} (1 - p_j - p_h)^{N-a_j-a_h} \cdot \frac{a_h}{N} \qquad (10.53)$$

Cancelling terms and readjusting the limits of summation of a_h and a_j, we find that

$p(A_h \text{ when } a_h < N - k)$

$$= \sum_{a_h=0}^{N-k-2} \binom{N-1}{a_h} p_h^{a_h+1} (1 - p_h)^{N-a_h-1}$$

$$- \sum_{j \neq h} \sum_{a_j=N-k}^{N-1} \sum_{a_h=0}^{N-a_j-1} \frac{(N-1)!}{a_j! a_h! (N-a_j-a_h-1)!}$$

$$\cdot p_j^{a_j} p_h^{a_h+1} (1 - p_j - p_h)^{N-a_j-a_h-1} \qquad (10.54)$$

The first expression in (10.54) equals

$$p_h \cdot \left(1 - \sum_{a_h=N-k-1}^{N-1} \binom{N-1}{a_h} p_h^{a_h} (1 - p_h)^{N-a_h-1} \right)$$

The second expression in (10.54) may be simplified by removing the factor p_h and applying the binomial theorem to give

$$p_h \sum_{j \neq h} \sum_{a_j=N-k}^{N-1} \binom{N-1}{a_j} p_j^{a_j} (1 - p_j)^{N-a_j-1}$$

Thus, (10.54) becomes

$p(A_h$ when $a_h < N - k)$

$$= p_h - p_h \sum_{a_h=N-k-1}^{N-1} \binom{N-1}{a_h} p_h^{a_h} (1 - p_h)^{N-a_h-1}$$

$$- p_h \sum_{j \neq h} \sum_{a_j=N-k}^{N-1} \binom{N-1}{a_j} p_j^{a_j} (1 - p_j)^{N-a_j-1} \qquad (10.55)$$

Substituting (10.15) and (10.55) into (10.50), we obtain

$$p(A_h) = \sum_{a_h=N-k}^{N} \binom{N}{a_h} p_h^{a_h} (1 - p_h)^{N-a_h}$$

$$+ p_h \cdot \left[1 - \sum_{a_h=N-k-1}^{N-1} \binom{N-1}{a_h} p_h^{a_h} (1 - p_h)^{N-a_h-1} \right.$$

$$\left. - \sum_{j \neq h} \sum_{a_j=N-k}^{N-1} \binom{N-1}{a_j} p_j^{a_j} (1 - p_j)^{N-a_j-1} \right] \qquad (10.56)$$

Now, in place of

$$\sum_{a_h=N-k}^{N} \binom{N}{a_h} p_h^{a_h} (1 - p_h)^{N-a_h} - \sum_{a_h=N-k-1}^{N-1} \binom{N-1}{a_h} p_h^{a_h+1} (1 - p_h)^{N-a_h-1}$$

we may write

$$\sum_{a_h=N-k}^{N} \left[\binom{N}{a_h} - \binom{N-1}{a_h-1}\right] p_h^{a_h} (1 - p_h)^{N-a_h}$$

which in turn equals

$$\sum_{a_h=N-k}^{N-1} \binom{N-1}{a_h} p_h^{a_h} (1 - p_h)^{N-a_h} \quad .$$

Substituting back into (10.56), replacing a_h and a_j by the single variable of summation, a, and rearranging terms, we obtain finally

$$p(A_h) = p_h + \sum_{a=N-k}^{N-1} \binom{N-1}{a} \left[p_h^a (1 - p_h)^{N-a} - p_h \sum_{j \neq h} p_j^a (1 - p_j)^{N-a-1} \right] \quad (10.57)$$

While expression (10.57) is quicker to compute, expression (10.56) affords greater insight into the behaviour of $p(A_h)$ when N becomes large. Every summation in (10.56) involves the upper section of a binomial distribution. Hence, by (3.6) as N becomes large, a summation with parameter p_i, any i, $i = 1, 2, \ldots, m$, approaches unity if $p_i > 1/2$, and diminishes to zero if $p_i < 1/2$. Thus, in expression (10.56), if $p_h > 1/2$ then as N becomes large both summations with parameter p_h approach unity while the others tend to zero, with the result that $p(A_h)$ also approaches unity. On the other hand, if $p_i < 1/2$, all i, $i = 1, 2, \ldots, m$, then as N becomes large all summations in (10.56) tend to zero, with the result that $p(A_h)$ approaches a limiting value of p_h.

<u>Majority with equiprobability</u>. The collective decision is the alternative, if one exists, which receives more than half of the first preference votes. If no such alternative

exists then the collective decision is equally likely to be any of the alternatives which have at least one vote.

Consider the case m = 3. When N is odd, every preference configuration which results in the non - emergence of a majority winner has the property: $a_j > 0$, $j = 1, 2, 3$. It follows that the probability that A_h is the winner under the majority with equiprobability scheme is

$$p(A_h) = p(a_h \geq N - k) + (1/3)\left(1 - \sum_{j=1}^{3} p(a_j \geq N - k)\right)$$

$$\text{when N is odd.} \qquad (10.58)$$

As before, $p(a_j \geq N - k)$, $j = 1, 2, 3$, may be evaluated by means of expression (10.15).

When N is even, three of the preference configurations which result in the non - emergence of a majority winner have a zero component. The three configurations are:

(i) $a_1 = 0$, $a_2 = N/2$, $a_3 = N/2$;

(ii) $a_1 = N/2$, $a_2 = 0$, $a_3 = N/2$;

(iii) $a_1 = N/2$, $a_2 = N/2$, $a_3 = 0$.

Under the equiprobability subscheme the contribution made to $p(A_h)$ by each of these configurations is the probability of the configuration multiplied by 0 if $a_h = 0$, and by 1/2 if $a_h > 0$. Thus, for alternatives A_h, A_g, and A_f, the contributions to $p(A_h)$ are :

(i) $\binom{N}{N/2} p_g^{N/2} \quad p_f^{N/2} \;.\; 0 = 0$

(ii) $\binom{N}{N/2} p_h^{N/2} \quad p_f^{N/2} \;.\; \frac{1}{2}$

(iii) $\binom{N}{N/2} p_h^{N/2} \quad p_g^{N/2} \;.\; \frac{1}{2}$

Employing the same formula for $p(A_h)$ as applies in case N odd, and correcting for the overinclusion or underinclusion of the above probabilities, we obtain

$$p(A_h) = p(a_h \geq N - k) + (1/3)(1 - \sum_{j=1}^{3} p(a_j \geq N - k))$$

$$+ \frac{1}{6} \cdot \binom{N}{N/2} (p_h^{N/2} p_f^{N/2} + p_h^{N/2} p_g^{N/2} - 2p_f^{N/2} p_g^{N/2})$$

when N is even. (10.59)

When $m > 3$, we may write

$$p(A_h) = p(a_h \geq N - k) + (1/m)(1 - \sum_{j=1}^{m} p(a_j \geq N - k))$$

$$+ C_m \qquad (10.60)$$

where C_m represents a correction for underinclusion and overinclusion of terms in the second expression on the right hand side of (10.60).

Unfortunately, in a preference configuration which results in the non-emergence of a majority winner, there may be up to $m - 3$ zero - valued a_j when N is odd, and up to $m - 2$ zero - valued a_j when N is even. Thus, the apparent simplicity of formula (10.60) becomes compromised by the necessary introduction of a rapidly expanding number of terms in C_m as m increases. As expressions for C_m in case $m > 3$ tend to be rather lengthy they will not be presented here.

However, as N becomes large the overall effect of C_m on $p(A_h)$ becomes negligible. Therefore, the asymptotic behaviour of $p(A_h)$ is governed by (10.60) with $C_m = 0$. As N becomes large, if $p_h > 1/2$ then $p(A_h)$ tends to unity, and if $p_j < 1/2$, all j, then $p(A_h)$ approaches a limiting value of $1/m$.

For small N, an approximation for $p(A_h)$ is obtained if C_m is omitted from (10.60). This expression is exact when $p_j = 1/m$, $j = 1, 2, \ldots, m$.

10.3 SOME IMPLICATIONS OF THE MODELS

The plurality and majority procedures are widely employed as explicit social decision schemes. They are formally embodied in the election systems of several countries and in the ways in which many committees conduct their business. This longstanding association with democratic tradition, combined with a simplicity of operation, would appear to make either of these schemes a natural way for a group to arrive at a decision. However, notwithstanding the prevalence of the plurality and majority procedures as explicit social choice methods, a review of empirical findings undertaken by Davis et al (1976) indicates that the implicit social decision schemes of groups vary greatly depending on such factors as membership abilities, task, social setting, etc.

The plurality, majority, Borda, and Condorcet schemes all predict a certain amount of equivocation; that is, they cater for the possibility that the group fails to reach a clear - cut decision. By contrast, the other schemes always predict a decisive outcome. Although group indecision seems not to occur in experimental studies this may be a function of task or instructions.

A number of the models predict that $p(A_h)$ remains constant over a given range of situations :

(a) given p_h, the proportionality model states that $p(A_h)$ remains constant, irrespective of variations in N, in m, and in the other p_j, $j \neq h$;

(b) given p_h and N, the majority model asserts that $p(A_h)$ keeps the same value, notwithstanding fluctuations in m and in the other p_j, $j \neq h$;

(c) given p_h and N, and provided A_h has either the highest or lowest expected value, the highest expected value model declares that $p(A_h)$ is unaffected by changes in m and in the other p_j, $j \neq h$;

(d) given p_h, p_g, and N, the majority if A_g proportionality otherwise model affirms that $p(A_h)$ remains constant in spite of changes in m and in the other p_j, $j \neq g$, $j \neq h$;

(e) given p_h, N odd, and m = 3, the majority with equiprobability model predicts that $p(A_h)$ remains unchanged as the other p_j, $j \neq h$, vary.

As N becomes large each of the social decision scheme models makes a different prediction about the limiting value of $p(A_h)$. Table 10.1 summarises the implications of each model in this respect.

An examination of Table 10.1 suggests a potentially powerful method by which an empirical comparison of the models might be achieved. Where feasible, the manipulation of the p_j by the experimenter could provide an effective and economical means of discriminating between the models when N is large. For example, let us suppose that the experimenter can arrange matters so that $p_h > p_j$, all $j \neq h$, and so that A_h does not have the highest expected value. In this event, Table 10.2 demonstrates in case m = 5 how to discriminate between seven models by performing two experiments. The probability p_h is arranged to equal .4 in the first experiment, and .6 in the

TABLE 10.1

PROBABILITY AS N BECOMES LARGE THAT A GROUP SELECTS ALTERNATIVE A_h, FOR EACH OF EIGHT SOCIAL DECISION SCHEME MODELS

MODEL	$p(A_h)$ as $N \to \infty$
PLURALITY	
(i) $p_h > p_j$, all $j \neq h$	1
(ii) $p_h < p_j$, some $j \neq h$	0
MAJORITY	
(i) $p_h > 1/2$	1
(ii) $p_h < 1/2$	0
PROPORTIONALITY	p_h
EQUIPROBABILITY	$1/m$
HIGHEST EXPECTED VALUE	
(i) $EV(A_h) > EV(A_j)$, all $j \neq h$, and $p_h > 0$	1
(ii) $EV(A_h) < EV(A_j)$, some $j \neq h$	0
MAJORITY IF A_g, PROPORTIONALITY OTHERWISE	
(i) $p_g > 1/2$	0
(ii) $p_g < 1/2$	p_h
MAJORITY WITH PROPORTIONALITY	
(i) $p_h > 1/2$	1
(ii) $p_j < 1/2$, all j	p_h
MAJORITY WITH EQUIPROBABILITY	
(i) $p_h > 1/2$	1
(ii) $p_j < 1/2$, all j	$1/m$

TABLE 10.2

PROBABILITY WHEN $m = 5$ AND N IS LARGE THAT A GROUP SELECTS A_h, FOR EACH OF SEVEN MODELS IN TWO EXPERIMENTAL CONDITIONS. IN BOTH EXPERIMENTS p_h IS EXPERIMENTER - DETERMINED SUCH THAT $p_h > p_j$, all $j \neq h$; ALSO A_h DOES NOT HAVE THE HIGHEST EXPECTED VALUE.

MODEL	EXPERIMENT 1 $p_h = .4$	EXPERIMENT 2 $p_h = .6$
PLURALITY	1	1
MAJORITY	0	1
PROPORTIONALITY	0.4	0.6
EQUIPROBABILITY	0.2	0.2
HIGHEST EXPECTED VALUE	0	0
MAJORITY WITH PROPORTIONALITY	0.4	1
MAJORITY WITH EQUIPROBABILITY	0.2	1

TABLE 10.2

PROBABILITY WHEN $m = 5$ AND N IS LARGE THAT A GROUP SELECTS A_h, FOR EACH OF SEVEN MODELS IN TWO EXPERIMENTAL CONDITIONS. IN BOTH EXPERIMENTS p_h IS EXPERIMENTER - DETERMINED SUCH THAT $p_h > p_j$, all $j \neq h$; ALSO A_h DOES NOT HAVE THE HIGHEST EXPECTED VALUE.

MODEL	EXPERIMENT 1 $p_h = .4$	EXPERIMENT 2 $p_h = .6$
PLURALITY	1	1
MAJORITY	0	1
PROPORTIONALITY	0.4	0.6
EQUIPROBABILITY	0.2	0.2
HIGHEST EXPECTED VALUE	0	0
MAJORITY WITH PROPORTIONALITY	0.4	1
MAJORITY WITH EQUIPROBABILITY	0.2	1

TABLE 10.2

PROBABILITY WHEN $m = 5$ AND N IS LARGE THAT A GROUP SELECTS A_h, FOR EACH OF SEVEN MODELS IN TWO EXPERIMENTAL CONDITIONS. IN BOTH EXPERIMENTS p_h IS EXPERIMENTER - DETERMINED SUCH THAT $p_h > p_j$, all $j \neq h$; ALSO A_h DOES NOT HAVE THE HIGHEST EXPECTED VALUE.

MODEL	EXPERIMENT 1 $p_h = .4$	EXPERIMENT 2 $p_h = .6$
PLURALITY	1	1
MAJORITY	0	1
PROPORTIONALITY	0.4	0.6
EQUIPROBABILITY	0.2	0.2
HIGHEST EXPECTED VALUE	0	0
MAJORITY WITH PROPORTIONALITY	0.4	1
MAJORITY WITH EQUIPROBABILITY	0.2	1

second experiment. Since no two models predict the same pattern of results across the two experiments, the best - fitting model, if there is one, will be apparent. Of course, this example oversimplifies matters somewhat : for feasible values of N, $p(A_h)$ may not yet be close to its limiting value in some models; also, it would be advisable to consider the behaviour of $p(A_j)$, all j, rather than simply the behaviour of $p(A_h)$. Nevertheless, where applicable, the fundamental strategy of manipulating the p_j would appear to provide a strong test of the relative merits of competing social decision scheme models.

Lastly, in the same vein, it is worth noting that the preference ordering probabilities, q_j, $j = 1, 2, ..., m!$, which permit maximal discrimination between the plurality and Condorcet schemes, are those contained in culture $q_{max}(A_i; m)$ where the likelihood of plurality - Condorcet disagreement is at its highest. (Culture $q_{max}(A_i; m)$ was determined in chapter 7.) Likewise, as established in chapter 8, when m = 3 culture q = (0, 5/12, 7/12, 0, 0, 0) provides an effective means of discriminating between the Borda and Condorcet schemes.

CHAPTER 11

CONCLUSION

The issues examined in the course of this study have profound ramifications which transcend disciplinary boundaries. The fundamental problem under investigation, viz. the aggregation of N orderings of m alternatives into a single ordering, or into a single alternative, is widely encountered in different guises in a variety of contexts. Moreover, the basic multinomial model employed in the present study is sufficiently general and realistic in its assumptions to be applicable and useful in most contexts. Consequently, the results obtained and the methods developed are of considerable interest to psychologists, mathematicians, economists, sociologists, and political scientists, amongst others.

Indeed, many of the probabilistic solutions achieved in the course of this investigation have long been sought after by researchers from a from a number of different disciplines. For example, in economics attention was first called to the need for an estimate of the likelihood of the paradox of voting by Black (1958) ; in psychology the apparent intractability of the general social decision scheme model was deprecated in a seminal article by Davis (1973) ; and in mathematics the possibility of a probabilistic treatment of the reversed-order type of paradox was adumbrated in the work of Davidson and Odeh (1972). In a few instances, probabilistic solutions have actually been obtained by previous researchers. However, in all such cases the results achieved in this thesis are considerably simpler and speedier to compute.

This inquiry has been concerned with two important questions both of which may be regarded as manifestations of the fundamental aggregation problem : (i) how to determine what a group thinks, and (ii) how a group itself decides what it thinks. These questions refer to apparently separate issues and have in the past generated independent areas of research. The former has led to the search for a suitable objective index of collective opinion, while the latter has focused attention on the more or less tacit methods by which informal groups actually arrive at a collective decision. The contrasting goals of these two areas of research reflect their different underlying philosophies, normative in the first case and descriptive in the second. Now, clearly one would not expect the outcome provided by an objective index to coincide always, or even often, with that produced by a tacit decision scheme. Group decisions are not always a fair reflection of member opinion. Nevertheless, what previous researchers have failed to appreciate fully is that these divergent areas of research with seemingly dissimilar goals share a common underlying structure. Results in one field have implications for the other field. Both may benefit from a cross-fertilisation of ideas.

The multinomial choice model has proved to be an excellent vehicle for such a conceptual interchange. Its assumptions are realistic enough for it to be relevant and serviceable in most disciplines. Thus, probability expressions obtained for normative purposes in economics may also be used for descriptive purposes in psychology. For example, the Condorcet and Borda procedures may be employed as social decision scheme models ; so that the solutions obtained for these formal collective choice procedures in Chapters 4 and 5 also succeed in answering Davis's (1973, 1976) repeated appeals for extensions of the general social decision scheme model which incorporate full preference orderings.

The blinkered compartmentalisation of the various approaches to the aggregation problem, and the attendant blindness of researchers in different fields to the formal similarities underlying their respective endeavours, has been nowhere more evident than in the social sciences. Here, the first question addressed by this thesis, viz. the normative issue of how to determine what a group thinks, has two separate, almost non-interacting, traditions of approach : (i) the social choice tradition, and (ii) the data analysis tradition. The former, exemplified by the work of Fishburn (1973) and Sen (1970), places great emphasis on the criteria which a good objective index of collective opinion should satisfy, while the latter, typified by much of the elementary scaling and statistical literature, e.g. Torgerson (1958), Winer (1971), is more concerned that an objective index should be reasonably straightforward to manipulate statistically. Thus, the social choice tradition pays due regard to problems like that of interpersonal comparability of judgmental intensity, whereas more often than not the data analysis tradition ignores this problem and consequently, by default, assumes complete interpersonal comparability of judgmental intensity. Now, the normative results of this thesis are relevant to both the social choice and data analysis traditions. Indeed, the plurality, Condorcet, and Borda collective choice rules were specifically selected for analysis not only because of their significance in terms of social choice theory but also because of their formal equivalence to common methods of data analysis. Thus, while investigating matters of great importance for democratic theory we are simultaneously advancing our understanding of popular techniques of data analysis. For example, the study of plurality-Condorcet disagreement in Chapter 7 enables us to determine the likelihood that an objective index of collective opinion derived from first choices only misrepresents the views of a group of individuals. Similarly, the study of the Borda reversed-order paradox in Chapter 9 provides a means of estimating the likelihood of inconsistencies, arising from the failure of assumptions like that of interpersonal comparability of judgmental intensity, in an objective index of group opinion derived from rank order data.

A primary concern of this thesis has been the influence of structural factors on collective choice, e.g. the number of alternatives and the number of group members. From the normative point of view it is easy to see why structural considerations should predominate. The normative problem is essentially structural in character. What then is the advantage of a structural emphasis when dealing with the descriptive problem? Clearly, when investigating group behaviour the effect of structure must be removed, or at least held constant, before the effect of non-structural factors can be established. The influence of non-structural variables can only be accurately determined once the influence of structure is known. For example, Davis (1973) argues that the well-known risky shift finding, in which a group's decision is less cautious than the mean of the decisions of its individual members, may be a consequence of a purely structural factor, viz. the implicit social decision scheme employed by the group. Explanations of this phenomenon in terms of psychological processes such as responsibility diffusion or familiarisation may therefore be superfluous.

From all that has been said so far, it is evident that a major benefit of the present probabilistic approach to the aggregation problem is that it provides a framework, or paradigm, which subsumes both the normative approach (whether in its social choice form or in its data analysis form) and the descriptive approach. Parenthetically, it is interesting to note that the generality of the framework extends beyond the topics specifically mentioned in this thesis. For example, as well as addressing the issues of how consensus is to be measured and how a group actually arrives at a consensus, the paradigm is also relevant to the question of how consensus is perceived at the level of the individual. That is, when a group of N members deliberates on m alternatives, what are the likelihoods which an individual with knowledge (accurate or otherwise) of that culture assigns to the possible outcomes ? To illustrate the advantages of the present conceptual framework in dealing with the aggregation problem , the implications and applications of the results of the present investigation are considered, firstly from the normative point of view and secondly from the descriptive point of view. Finally, some directions for future research are suggested.

NORMATIVE IMPLICATIONS

The bearing of the findings of this inquiry on the search for suitable objective methods of aggregating diverse individual judgments is examined from (a) the social choice perspective and (b) the data analysis perspective.

(a) Social choice perspective. It has long been recognised (Sen, 1970) that it is very difficult to obtain an accurate cardinal measure of the relative intensity with which preferences are held by a group as a whole. Individuals have a vested interest in the outcome of collective decision making and are liable to misrecord or distort the intensity of their preferences for the various alternatives. Nevertheless, were knowledge of the sum of group members' true preference intensities in respect of each alternative miraculously revealed to us, it is extremely unlikely that any two alternatives would obtain precisely the same sum. Now, collective choice rules such as the plurality, Condorcet and Borda procedures almost invariably operate at the ordinal, rather than the cardinal, level. In so doing, they avoid some of the problems produced by preference distortion but pay a price in that precision of measurement is sacrificed. Collective indecision, unlikely in terms of the true strength of group feeling, becomes a real possibility because of the limitations of our collective choice rules. Now, it can be argued that decisiveness is a desirable attribute of a collective choice procedure. Thus, Niemi and Riker (1976) state that "the minimum we want voting to accomplish is a clear-cut decision". A detailed comparison of the plurality, Condorcet and Borda procedures in terms of their relative susceptibility to indecision has been provided in Chapter 6.

The plurality, Condorcet, and Borda procedures differ in many respects other than in decisiveness. Therefore, unless these differences are considered unimportant by the group concerned, it cannot simply be argued that the most decisive of the three should be adopted in a given situation. In general, however, should a number of decision procedures satisfy the criteria required of a collective choice rule by a group, and should each rule prove to be most decisive in a different situation, e.g. group size, then instead of adopting a particular one of these procedures to be used on every occasion, a group might employ all of them at one time or another. The procedure chosen on a given occasion would be the one which is least equivocal in that context. It is quite likely, as was found in the comparison of the plurality, Condorcet and Borda procedures, that in the majority of settings no procedure is consistently most decisive. In this event, a group might choose the procedure which minimises the maximum likelihood of indecision.

One problem arising from the adoption of such a minimax strategy should be mentioned. It is possible that procedure A is more decisive than procedure B across the vast majority of cultures and less decisive in only a few cultures. In this event, it could happen that in these few cultures procedure A has a likelihood of indecision which is greater than that experienced by procedure B in any culture. That is, although procedure A is more decisive in most cultures, procedure B has a smaller upper limit for the likelihood of indecision. A minimax strategy would therefore favour procedure B when in fact procedure A is the better choice.

Whether or not decisiveness proves a useful criterion in the selection of a collective choice procedure , information about the probability of collective indecision is useful for groups which are constitutionally committed to a particular collective choice procedure. Knowledge of those situations, e.g. group sizes, where the probability of deadlock in the decision making process is at its highest and lowest values, is advantageous for the group. For example, information on the upper limit of the probability of indecision for given N and m would enable the size of a committee's membership to be determined so as to minimise the maximum likelihood of indecision. That is, when a minimax strategy cannot be used to select a suitable collective choice procedure, since that has already been determined, it can still be used to select a suitable group size. Detailed recommendations on group size for each of the plurality, Condorcet, and Borda procedures have been supplied at the end of Chapters 3, 4, and 5. More importantly, formulae have been provided in each of these chapters so that any group may easily calculate the probability of indecision in its own special situation.

Turning our attention to the problems associated with the plurality and Borda procedures, which were investigated in Chapters 7, 8, and 9, we find that a number of conclusions may be drawn. The plurality procedure has grave shortcomings which arise because it considers only first preferences. In effect, it asks the following question. What alternative, if any, has most first preference support? The Condorcet procedure utilises more of the ordinal information in an individual's preference ordering and is widely held to be superior to the plurality procedure (Fishburn, 1973). Now, we have shown that the likelihood of plurality - Condorcet disagreement is unacceptably high in many cultures. Of special interest is the case of the group or society which is polarised along a single dimension, e.g. left wing/right wing, Arab/Jew, Catholic/ Protestant. In such single-peaked cultures, if the likelihood that an individual has a "moderate" first preference is low, the probability of plurality - Condorcet disagreement is at its maximum. By thwarting the aspirations of the majority in this way, the use of the plurality procedure can be one of the factors contributing to group disharmony and social unrest.

Like the Condorcet procedure, the Borda procedure utilises much of the ordinal information in an individual's preference ordering. However, this information is used for different ends by the two procedures. The Condorcet procedure asks the following question. Do more people prefer A_i to A_j than the other way round ? The Borda procedure asks the following question. Is the average intensity of preference greater for A_i than for A_j ? The Condorcet procedure aims to establish the will of the majority, while the Borda procedure is prepared to allow a minority with strong feelings to outweigh a majority with mildly opposite feelings. Whereas the Condorcet procedure makes no assumptions about an individual's preference ordering other than that it represents a linear ordering of the alternatives, the Borda procedure assumes both that a preference ordering constitutes cardinal measurement of preference intensity for the alternatives and that there is interpersonal comparability of preference intensity.

Broadly speaking, we have shown that the Borda procedure is at least as likely as the plurality procedure to produce an outcome which is in agreement with that selected by the Condorcet procedure, i.e. which accords with the wishes of the majority. In this respect, then, the Borda procedure is superior to the plurality procedure. However, the Borda reversed-order paradox, where removal of an alternative leads to an inversion of the relative positions of other alternatives in the social ordering, turns out to have a probability of occurrence which is surprisingly high. Even in cultures where the paradox of voting cannot arise, i.e. in value-restricted cultures, the likelihood of the Borda reversed-order paradox can be considerable. Therefore, to the extent that the occurrence of a Borda reversal is regarded as a deficiency of similar gravity to the occurrence of a cyclical majority, the Borda procedure would appear to compare unfavourably with the Condorcet procedure.

(b) Data analysis perspective. Not all the criteria which have to date been proposed for collective choice procedures will be equally relevant in both the social choice domain and the data analysis domain. For example, indecision is an embarassment in general elections but informative in market research. On the other hand, an examination of some of the untested assumptions of popular techniques of data analysis is made all the more instructive when the basic similarity of these techniques to common methods of social choice is recognised. Much elementary scaling in the social sciences is carried out quite uncritically by researchers using time-honoured procedures which have acquired an aura of scientific respectability by dint of frequent usage. Such procedures are employed sometimes in the preparatory stages of an investigation, e.g. calibrating the perceptual complexity of a number of stimuli prior to an experiment, and sometimes in the main part of an investigation, e.g. determining preferences for various brands of a product. Whether an elementary scaling procedure is used as a means to an end or as an end in itself, the object of the exercise is often to pool the judgments of a group of individuals in order to achieve a rank ordering of the stimuli along a particular dimension by the group as a whole. (Frequently, only the stimulus regarded by the group as a whole as most, or least, intense in terms of this dimension is required.) However, this apparently clear-cut goal actually admits of several interpretations. The source of the ambiguity lies in the phrase "the group as a whole". Researchers often fail to consider carefully which sense of the expression "the group as a whole" is appropriate in a given context. They are frequently unaware that the various methods of aggregating individual judgments address themselves to different aspects of the data and, in effect, answer different questions about the data. For example, one method might seek to establish whether a majority of individuals considers A_i to be more intense than A_j, while another might attempt to determine whether the average intensity associated with A_i is higher than that associated with A_j.

Let us examine in more detail some of the implications of the present study for measurement procedures in the social sciences. In particular, let us consider the topics of (i) questionnaire design, (ii) inter-observer agreement, (iii) assumption failure in the popular Borda-type analysis of rank order data, and (iv) misconceptions about the binomial test.

(i) Questionnaire design. The multiple choice format is widely employed in the construction of questionnaires. For example, a factory worker might be asked :

Which of the following aspects of your present employment are you least satisfied with :

(a) your particular job,

(b) the surroundings in which you work,

(c) the factory management,

(d) your wages ?

The overall opinion of factory workers is generally taken to be that aspect, if one emerges, which receives most endorsement. In other words, the multiple choice format leads naturally and, one might add, almost inevitably to a plurality-type analysis. Now, as we have demonstrated in our examination of plurality-Condorcet disagreement in Chapter 7, the plurality procedure frequently misrepresents the opinions of the majority. Moreover, much better methods of assessing the views of the majority (if that is what the researcher wants to measure) are available, e.g. the Condorcet procedure. Why, then, should the plurality procedure remain so popular ? Although its simplicity of operation is clearly an advantage, two other factors are probably largely responsible for its continued widespread use. First, the plurality procedure's long-standing association with fundamental democratic processes, e.g. the election of members of parliament, has conferred upon it an image of fairness and representativeness. Second, social scientists mistakenly reason that, when they are looking for the first preference of a group as a whole, they need consider only the first preferences of the individual members. Whatever the origins of the social scientist's firmly rooted attachment to the plurality procedure, it is evident that the aims of much scientific inquiry would be better served by requiring respondents to questionnaires, where appropriate, to rank order the alternatives in a multiple choice question in terms of the dimension of interest. Unless strong practical or theoretical considerations dictate otherwise, a researcher should not restrict subjects' responses to first choices (i.e. most, or least, intense judgments).

(ii) Inter-observer agreement. In the social sciences, observers are often trained to assess individuals, situations, behaviours, etc., by either (a) categorising, or (b) rank ordering, them in terms of a particular dimension. For example, in an experiment on group dynamics each of a number of observers might be asked (a) to nominate the member who best fits the description of "group leader", or (b) to rank order the group members in terms of leadership ability. The experimenter is generally interested not only in the aggregate assessment of the observers but also in the extent to which the observers concur in this aggregate assessment. Typically, in a version of the method of categorisation where the observers state the individual, situation , behaviour, etc., that pre-eminently possesses a particular property, a plurality-type procedure is employed to determine both the aggregate assessment itself and the degree of inter-observer agreement. From the analysis in Chapter 7, it is clear that unless substantially more than N/2 out of N observers choose the same individual, situation, behaviour, etc., the plurality index can be a poor, and often misleading, measure both of collective observer opinion and of inter-observer agreement.

On the other hand, when the rank order method is employed the aggregate assessment is commonly arrived at by a Borda-type procedure while the degree of inter-observer agreement is provided by an index such as Kendal's coefficient of concordance. For the relationship between the Borda outcome and the coefficient of concordance the reader should consult Winer (1971, pp. 301-303). Now, the findings contained in Chapter 9 of this thesis are significant in the present context. They demonstrate how even in circumstances where inter-observer agreement can be expected to be reasonably high our confidence in the representativeness and consistency of the Borda outcome may be somewhat less than complete. To be more precise, let us consider three factors which tend to promote inter-observer agreement. First, agreement is more likely in a group of observers who all share the same culture (training) than in a group of observers who each belong to different cultures, given that the shared culture of the first group of observers is the mean of the diverse cultures of the second group of observers. In this respect, the assumption by the multinomial choice model that a uniform culture prevails implies that, if anything, the findings of Chapter 9 err on the side of caution. If anomalies in the Borda outcome can arise with a group of observers all sharing the same culture then a fortiori these anomalies will be likely to occur with a group of observers each from radically disparate cultures whose mean is the shared culture of the first group. Second, agreement is more likely if observers view the alternatives in terms of the same unitary dimension, i.e. if the culture is single-peaked. Third, agreement is more likely when there exists a specific preference ordering with a reasonably high probability of being chosen by an observer. Now, even when all of these three factors which tend to promote inter-observer agreement operate simultaneously, i.e. when the observers all share a single-peaked culture in which a specific preference ordering has a fairly high probability of being selected, the analysis of the Borda reversed-order paradox in Chapter 9 demonstrates that inconsistencies and anomalies in a Borda-type aggregate assessment can still have a disconcertingly high likelihood of occurrence. Thus, even a reasonably high level of inter-observer agreement does not guarantee the representativeness of the Borda outcome.

(iii) Assumption failure in the popular Borda-type analysis of rank order data. The reversed-order paradox arises when two crucial assumptions underlying the Borda procedure are violated, viz. that a preference ordering constitutes cardinal measurement of preference intensity for the alternatives, and that there is interpersonal comparability of preference intensity. The investigation of the reversed-order paradox in Chapter 9 has enabled us to examine thoroughly the conditions under which this particularly disturbing consequence of assumption violation can occur, and to calculate the likelihood of its occurrence in a given set of circumstances, e.g. culture, group size, number of alternatives.

(iv) Misconceptions about the binomial test. The study of Borda-Condorcet disagreement in Chapter 8 highlights the flaws in the belief, common among social scientists, that a binomial test of significance may be regarded as essentially a weaker, cruder version of a t-test, or of a Friedman multiple comparison (between two levels of a factor with more than two levels; a Friedman test on a factor with only two levels is equivalent to a binomial test). Even when assumptions such as interpersonal comparability of judgmental intensity and cardinal measurement of judgmental intensity hold, the binomial test need not produce the same result as a t-test or a Friedman multiple comparison. The binomial test asks whether a majority of individuals considers A_i to be more intense than A_j, whereas the t-test and Friedman test ask whether the average judged intensity of A_i is greater than that of A_j.

DESCRIPTIVE IMPLICATIONS

Social decision scheme theory has exercised a considerable influence on research on collective decision making. In a review of topics which have been investigated from the standpoint of social decision scheme theory, Davis (1976) mentions, amongst others, the following : the strategy preferences of groups performing a sequential response task, choice shift effects, and the influence of group size and member ability on group performance on an intellective task. However, although the theory has been widely applied, its implications and predictions have hitherto only been obtainable by the technique of computer enumeration. Closed-form mathematical solutions have previously been regarded as impossible to achieve (Davis, 1973). One of the contributions of this thesis has been to show, in Chapter 10, that this view was unduly pessimistic. Moreover, not only have formulae been derived for the current version of the model which incorporates first preferences only, but the expressions developed in Chapters 4 and 5 in connection with the Condorcet and Borda procedures provide formulae for an extended version of the model embracing full preference orderings.

Social decision scheme theory focuses mainly on the structural aspects of collective decision making. One justification for such an approach is that the importance of non-structural variables can only satisfactorily be assessed once the contribution of structural factors is known. Two structural variables which may influence the course of group decision making are N, the number of group members, and m, the number of alternatives. Outlining his general social decision scheme model, Davis (1973, p. 121) makes the point that group size N, has been a relatively neglected variable in psychological research because hitherto "the meaning of group size has not been anchored in theory as an interpretable parameter with exact consequences". Social decision scheme theory redresses the balance by incorporating both N and m as explicit parameters.

Ultimately, to establish whether structural variables, like N and m, have an effect on social interaction in decision-making groups, empirical investigation is required. However, the availability of a theory embodying structural variables helps considerably by providing guidance and motivation for empirical research. For example, Davis (1973) suggests that there could be a relationship between structural factors and the amount of internal conflict in a group. He postulates that feelings of disunity and disharmony will be especially prevalent among the members of a group which is split into equal-sized factions each supporting different alternatives. Since social decision scheme theory is able to predict those values of N and m which are most, or least, likely to generate tied subgroups, we are in a position to construct a strong empirical test of the influence of N and m on the amount of internal conflict in a group.

Now, as Davis (1973, p. 123) points out, "social decision scheme theory is not, of course a theory of all group decision making". Important aspects of collective choice which are not encompassed by the general model in its current form include : (a) subcultural diversity, i.e. the possibility of more than one subculture within the same group ; (b) time-dependent dynamic processes : the social processes intervening between the initial expression of member preferences and the final arrival at a collective decision are condensed by the model into the d_{ij} values, so that dynamic aspects of social interaction such as the influence of persuasive personalities and the formation of coalitions are excluded from consideration ; (c) strategic voting : an implicit assumption of the model is that a member's choice reflects his actual preference ; (d) previous group decisions : connections between group decisions made on separate occasions, e.g. through bargaining or vote-trading, are not accomodated by the model.

Whether or not the above omissions are serious enough to restrict the applicability of the model depends largely on the specific field under investigation. Certainly, some of the points raised do not constitute inherent limitations of the model. For example, the model may readily be extended to encompass subcultural diversity. The author has derived probability expressions for several special cases of just such an extended version of the model. Of course, the number of parameters to be estimated increases rapidly. Since we have already extended the model to include full preference orderings, we have m! in place of m cultural probabilities. Should each member belong to a different subculture, the number of cultural probabilities rises to N.(m!). Thus, although there are several directions in which the model might be refined and extended, inevitably a balance has to be struck between psychological realism on the one hand and mathematical tractability on the other. As it stands, the general social decision scheme model is by far the most realistic of its kind in social psychology. Indeed, its nearest rival, the Lorge-Solomon Model A (1955), was shown in Chapter 10 to be a special case of the general model.

DIRECTIONS FOR FUTURE RESEARCH

The method employed in the present study to investigate decisiveness represents a new approach to the normative analysis of collective choice rules. The strict conditional approach mentioned in Chapter 1 simply considers whether or not a criterion is satisfied by a particular collective choice rule. However, failure to satisfy a criterion may be a rare or a commonplace event. That is, criteria which collective choice rules are unable to meet can still provide information on important ways in which these rules differ from one another. Essentially, the method adopted in the present analysis is a combination of the conditional and likelihood approaches discussed in Chapter 1. When a collective choice rule is unable to satisfy a criterion, the likelihood with which it violates the criterion is determined and used for comparative purposes. Now, there are numerous criteria which many collective choice rules fail to meet. Richelson (1975) mentions five such criteria. Therefore, a useful direction is which future research might proceed is to extend the present approach to criteria other than decisiveness.

The current form of the multinomial choice model assumes that all group members belong to the same culture. As mentioned in connection with social decision scheme theory, the model may readily be elaborated to incorporate subcultural diversity among the members. Of course, considerations of parsimony demand that we first investigate the fit of the model which assumes cultural homogeneity. However, preliminary investigation by the author suggests that the question of subcultural diversity may not be so important or problematic as at first sight appears to be the case. It would seem that the operation of a form of central limit theorem ensures that, when N is large, predictions based on cultural homogeneity do not differ substantially from predictions based on subcultural diversity. However, further exploration of this issue is required.

Finally, this study has provided a general framework for conceptualising the effect of structural variables on the outcome of the aggregation process. It is hoped that the results and techniques developed in this work will stimulate further examination of both normative and descriptive aspects of this area.

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