SOME ASPECTS OF B-COMPLETENESS AND THE CLOSED GRAPH THEOREM

by

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ABSTRACT

One of the deepest results of Functional Analysis is the closed graph theorem of Banach which was originally proved for Banach spaces and has by now been generalised in many ways. Our aim in this thesis is to give some new forms of the closed graph theorem, describe some of its applications and find out more about B-complete spaces and algebras.

Chapter I of the thesis deals with preliminary materials. In Chapter II we give a new and simpler proof of a result of Šavgulidze and Smoljanov which asserts that $E \times \phi$ is B-complete whenever E is a B-complete locally convex space and ϕ is the topological direct sum of countably many copies of the scalar field. In our proof we have avoided explicit use of dual spaces, which allows us to extend our results to certain non-locally convex situations and topological algebras. We discuss non-locally convex cases in Section 3.

In Chapter III we introduce classes $\mathcal{R}$ and $\mathcal{D}$ of topological vector spaces which serve as range and domain spaces for a closed graph theorem. We assume that $\mathcal{R}$ contains $F \times \phi$ for every $F \in \mathcal{R}$ and prove that $\mathcal{D}$ contains each countable codimensional subspace of each of its elements. Using this fact along with the results of Chapter II we are able to extract some of the known results and some new results on the inheritance of topological vector space properties by subspaces of countable codimension. Also in this chapter, we give a closed graph theorem for webbed spaces.

In Chapter IV we have defined a new class of locally convex spaces called $M(\alpha)$-barrelled spaces. These spaces serve as domain

space for a closed graph theorem where the range space is an α-weakly compactly generated Banach space. The connection between this class of locally convex spaces and the class of domain spaces for a closed graph theorem with range space a Banach space of density character at most α is considered.

In Chapter V we are concerned with B- and B_r-completeness in locally convex algebras and obtain various extensions of our $E \times \phi$ results in this setting. In particular in Section 2 we looked at the possibility of B- or B_r-completeness of $E \times \phi$ when it has the usual pointwise multiplication. Similar work has been carried out in Sections 3 and 4 for the unitization of a B-complete or a B_r-complete algebra E and also for $E \times \phi$ when it has a new multiplication which is a natural generalization of the unitization.

CHAPTER I

PRELIMINARIES

§1. Introduction

The notions which have been collected together in this chapter are mostly well-known. The proofs, if needed, are easily available in any standard book on topological vector spaces. The results of Section 5 are relatively recent and do not appear in text books. The only new result in this chapter is Lemma 5.2.

§2. Notations

We will denote by $\mathbb{K}$ the scalar field $\mathbb{R}$ of real numbers or $\mathbb{C}$ of complex numbers. Our topological vector spaces and algebras are always over $\mathbb{K}$. When several topological vector spaces occur in one statement and no explicit mention of the respective scalar fields is made, the spaces involved are assumed to be defined over the same field $\mathbb{K}$, where either $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. The topological vector space (algebra) E with topology ξ will be denoted by $E[\xi]$ if it is necessary to name the topology. The term locally convex topological vector space will be contracted to locally convex space and our locally convex spaces (algebras) are always Hausdorff, although we do not require this all the time; e.g. most of our material in Chapter III is independent of Hausdorffness.

E^* will denote the algebraic dual of a vector space E and if E has a vector space topology ξ, by the dual of E we mean the topological or continuous dual of $E[\xi]$ and it will be denoted by E'. For a locally convex space E with dual E', the weak, Mackey and strong topologies of the dual pair (E, E') will be denoted by $\sigma(E, E')$, $\tau(E, E')$ and $\beta(E, E')$ respectively.

If A is a subset of a topological vector space $E[\xi]$ and t is a mapping of E into another space, the topology induced on A by $E[\xi]$ will be denoted by $\xi|_A$ and the restriction of t on A will also be denoted by $t|_A$. A° will denote the (absolute) polar of A in E^* .

We shall use card (I) for the cardinality of a set I . By density character of a topological space E we mean the least cardinal number for which E has a dense subset of this cardinality.

We will denote the origin of a vector space by 0 and the completion of a topological vector space E will be denoted by $\hat{E}$. For two sets A, B by $A \subseteq B$ we mean "A is a subset of B" and by $A \subset B$ we mean "A is a proper subset of B" . We will write $\text{cl}\, A$ for the closure of a subset A of a topological space.

For a subset M of a vector space E we will denote by $\Gamma(M)$ the absolutely convex envelope of M , i.e. the set of all finite linear combinations $\sum_i \lambda_i x_i$ with $\sum_i |\lambda_i| \leq 1$, $\lambda_i \in \mathbb{K}$ and $x_i \in M$ for each i . If $M = \bigcup_{i\in I} M_i$ we will write $\Gamma(M) = \Gamma_i M_i$ instead of $\Gamma(\bigcup_{i\in I} M_i)$.

If X is a topological space, the space of all real or complex valued continuous functions on X is denoted by $C(X)$. If E and F are topological vector spaces (or topological algebras) then $E \simeq F$ means E is topologically isomorphic with F . For an algebra E with multiplication . we shall write $(E, .)$. We shall mark the end of proofs by Δ .

§3. Topological vector spaces

We recall that a topological vector space E is a vector space over $\mathbb{K}$ with a compatible topology, i.e. with a topology ξ on E such that the algebraic operations in E are continuous. Every topological vector space has a base of balanced (and closed) neighbourhoods of 0 . If it happens that a topological vector space has a base of convex neighbourhoods of 0 then it is called a locally convex topological vector space; as we have mentioned before we contract this to locally convex space.

We shall use the following Theorem frequently:

3.1 Theorem ([37], Ch. I, Theorem 2)

A locally convex space E has a base $\mathcal{U}$ of neighbourhoods of 0 with the following properties:

(i) if $U \in \mathcal{U}$, $V \in \mathcal{U}$, there is a $W \in \mathcal{U}$ with $W \subseteq U \cap V$;

(ii) if $U \in \mathcal{U}$ and $\lambda \neq 0$, $\lambda U \in \mathcal{U}$;

(iii) each $U \in \mathcal{U}$ is absolutely convex and absorbent.

Conversely, given a non-empty set $\mathcal{U}$ of subsets of a vector space E with the properties (i), (ii) and (iii), there is a topology making E a locally convex space with $\mathcal{U}$ as a base of neighbourhoods of 0 .

Generally, we follow the notations and definitions of [37] except that by a Mackey space E we mean a locally convex space endowed with its Mackey topology $\tau(E, E')$.

One of the important problems which always has been considered in functional analysis is to look for the inheritance of the properties of spaces by their subspaces. In this aspect if we are working on topological vector spaces and a property of a certain space is not inherited by all of its subspaces, we are naturally led to consider special subspaces such as closed, finite codimensional, countable codimensional and so on. In this thesis we are more interested in the properties which are inherited by subspaces of countable codimension.

In connection with the closed graph theorem the following Lemma, which is contained in the proof of the Theorem in [6] and also in Proposition 2.1 of [18], will be useful.

We recall that the graph of a mapping t from a space E into another space F is the subset $\{(x, t(x)) : x \in E\}$ of $E \times F$, and will be denoted by $\mathcal{G}(t)$. If E, F are vector spaces and t is linear, then $\mathcal{G}(t)$ is a vector subspace of $E \times F$.

3.2 Lemma

Let E, F be topological vector spaces. Let G be a proper subspace of E and let t be a linear mapping of G into F such that $\mathcal{G}(t)$ is closed in $G \times F$ but not closed in $E \times F$. Then the closure of $\mathcal{G}(t)$ in $E \times F$ is the graph of a linear mapping t_1 which extends t to a domain D with $G \subset D \subseteq E$.

Proof

Let $D = \{x \in E : (x, y) \in \text{cl}\, \mathcal{G}(t)$ for some $y \in F\}$. Clearly D is a vector subspace of E which contains G. Define the mapping $t_1 : D \to F$ by $t_1(x) = y$ whenever $(x, y) \in \text{cl}\, \mathcal{G}(t)$. t_1 is well-defined; for if (x, y_1), $(x, y_2) \in \text{cl}\, \mathcal{G}(t)$ then, since the closure of a subspace is a subspace, $(x, y_1) - (x, y_2) = (0, y_1 - y_2) \in \text{cl}\, \mathcal{G}(g)$. Since $\mathcal{G}(t)$ is closed in $G \times F$ we have $\mathcal{G}(t) = G \times F \cap \text{cl}\, \mathcal{G}(t)$ and from $(0, y_1 - y_2) \in G \times F$ we get $(0, y_1 - y_2) \in \mathcal{G}(t)$. Thus

$y_1 - y_2 = t(0) = 0$ i.e. $y_1 = y_2$. Linearity of t_1 follows from the fact that cl $\mathcal{g}(t)$ is a subspace of $E \times F$. Since $\mathcal{g}(t)$ is not closed in $E \times F$, we have $G \subset D \subseteq E$. And finally it is clear that $t_1|_G = t$ and $\mathcal{g}(t_1) = \text{cl}\ \mathcal{g}(t)$ is closed in $E \times F$. Δ

§4. Finest locally convex topology

Let E be a vector space and I a non-empty set. For each $i \in I$ let E_i be a locally convex space and f_i a linear mapping of E_i into E so that $\bigcup_{i \in I} f_i(E_i)$ spans E . The inductive topology on E with respect to the family $\{(E_i, f_i) : i \in I\}$ is the finest locally convex topology for which each of the mappings $f_i (i \in I)$ is continuous from E_i into E . A neighbourhood base of 0 for this topology is given by the family $\mathcal{U}$ of all absolutely convex subsets U of E such that for each $i \in I$, $f_i^{-1}(U)$ is a neighbourhood of 0 in E_i . Such a base can be obtained by forming all sets of the form $\Gamma_i f_i(V_i)$, where V_i is any member of a neighbourhood base $\mathcal{V}_i$ of 0 in E_i .

Let $\{E_i : i \in I\}$ be a family of locally convex spaces over $\mathbb{K}$. The algebraic direct sum $\bigoplus_{i \in I} E_i$ of the family $\{E_i : i \in I\}$ is the subspace of $\prod_{i \in I} E_i$ consisting of all those elements with only a finite number of non-zero coordinates. Denote by $f_i (i \in I)$ the injection map $E_i \to \bigoplus_{i \in I} E_i$. The space $\bigoplus_{i \in I} E_i$ equipped with the inductive topology defined by the family $\{(E_i, f_i) : i \in I\}$ is called the <u>topological direct sum</u> of $\{E_i : i \in I\}$. This topology, called the direct sum topology, is finer than the topology induced on $\bigoplus_{i \in I} E_i$ by $\prod_{i \in I} E_i$; they coincide only in finite direct sums. A neighbourhood base of

0 in $\underset{i\in I}{\oplus} E_i$ is provided by all sets of the form $U = \Gamma_i f_i(V_i)$ where $\{V_i : i \in I\}$ is any family of respective neighbourhoods of 0 in the spaces E_i .

Our special interest is when each E_i is the scalar field $\mathbb{K}$ and I is countable. We denote the topological direct sum of countably many copies of the scalar field by ϕ . In fact ϕ is the space of all scalar sequences with finitely many non-zero coordinates. It is easy to see that a neighbourhood base at 0 in ϕ is given by the family of all sets

$$\bigcup_{m=1}^{\infty} \sum_{n=1}^{m} J(r_n)$$

where $J(r_n) = \{\lambda e_n : |\lambda| \le r_n\}$ for some sequence $(r_n)_{n \in \mathbb{N}}$ of positive numbers while for each $n \in \mathbb{N}$, e_n is the element of ϕ with 1 in the nth position and zero elsewhere. Also the topology of ϕ is determined by the seminorms

$$P_{(r_n)}((\lambda_n)) = \sum_{n=1}^{\infty} r_n|\lambda_n| \; , \; (\lambda_n) \in \phi \; ,$$

where (r_n) runs through all sequences of positive real numbers.

From the properties of the finest locally convex topology and topological direct sums we deduce the following.

(a) $\phi' = \phi^* = \prod_{n=1}^{\infty} \mathbb{K}_n \; ; \; (\mathbb{K}_n = \mathbb{K} \; , \; n \in \mathbb{N})$.

We will denote $\prod_{n=1}^{\infty} \mathbb{K}_n$ by ω .

(b) Each subspace of ϕ is closed. Every algebraic direct sum decomposition of ϕ is topological.

(c) ϕ is barrelled, being an inductive limit of barrelled spaces, but not metrisable. Also ϕ is B-complete (see Chapter II, Definitions 2.3), because it is the dual of the Fréchet space ω , ([37] Ch. VI, Supplement (1)).

(d) ϕ is separable and $\aleph_0$ - WCG but not WCG (see Section 5 for definitions).

Every vector space E can be identified with a direct sum of copies of $\mathbb{K}$, one for each member of a basis. The direct sum topology, the finest inducing the usual topology on each copy of $\mathbb{K}$, is therefore the <u>finest locally convex topology</u> on E. This topology is $\tau(E, E^*) = \tau(E, E')$ for which a base of neighbourhoods of 0 is the set of all absolutely convex absorbent subsets.

The family of all ~~balanced absorbent subsets~~ strings (Ch. II, section 3) of a vector space E defines a base of neighbourhoods of 0 for a linear topology on E called <u>finest linear topology</u>. Also the family of all balanced semiconvex (Ch. II, section 3) absorbent subsets of E defines a base of neighbourhoods of 0 for a semiconvex topology on E called <u>finest semiconvex topology.</u> On a vector space E, the finest locally convex topology is coarser than the finest semiconvex topology and the finest semiconvex topology is coarser than the finest linear topology, the three topologies coinciding when the dimension of E is at most countable([2], §1(6)).

§5. <u>G(α)-barrelled and α-WCG spaces</u>

In this section α is an infinite cardinal number.

The concept of G(α)-barrelledness is due to J. Popoola and I. Tweddle [32]. Let E be a locally convex space and let B be a barrel in E. If $q : E \to E/\bigcap_{\lambda>0}\lambda B$ is the quotient map, the gauge of q(B) is a norm on $E/\bigcap_{\lambda>0}\lambda B$. We shall say that B is a <u>G(α)-barrel</u> in E if $E/\bigcap_{\lambda>0}\lambda B$ has density character at most α for the resulting norm topology and that E is <u>G(α)-barrelled</u> if each G(α)-barrel in E is a neighbourhood of 0.

Note that a $G(\alpha)$-barrelled space remains $G(\alpha)$-barrelled under any finer topology of the same dual pair; also if β is an infinite cardinal number with $\alpha < \beta$, then each $G(\beta)$-barrelled space is $G(\alpha)$-barrelled.

M. Valdivia [50] calls a locally convex space E <u>α-barrelled</u> if each $\sigma(E', E)$-bounded set with cardinality at most α is equicontinuous. Each barrelled space is α-barrelled and each α-barrelled space is $G(\alpha)$-barrelled, but the reverse implications do not hold in general (see [32], Remarks 1, page 251 and [32] Examples (i)).

It is shown that:

<u>5.1 Theorem ([32] Theorem 3)</u>

A locally convex space E is $G(\alpha)$-barrelled if and only if, whenever F is a Banach space of density character at most α and $t : E \to F$ is a linear mapping whose graph is closed in $E \times F$, then t is continuous.

<u>Remark</u>

It is easy to see that Banach space may be replaced by B-complete space or B_r-complete space in Theorem 5.1.

As we shall see in Chapter IV, locally convex spaces which are generated by weakly compact subsets can be used as a range space for some closed graph theorems.

A locally convex space E is said to be <u>weakly compactly generated (WCG)</u> if there exists an absolutely convex weakly compact total subset A of E. A locally convex space E is said to be <u>α-weakly compactly generated (α-WCG)</u> if there is a family

$\{A_i : i \in I\}$ of weakly compact absolutely convex subsets of E such that $\mathrm{card}(I) = \alpha$ and $\bigcup\{A_i : i \in I\}$ is total (or equivalently, dense) in $E[\sigma(E, E')]$.

Clearly every locally convex space with density character at most α is α-WCG and every reflexive Banach space is WCG . In general, WCG and α-WCG spaces are different. For example, the space ϕ is $\aleph_0$-WCG but not WCG . But an $\aleph_0$-WCG Fréchet space is WCG ([15], Proposition 1.1).

The following Lemma, which we shall need later, is concerned with α-WCG spaces and $G(\alpha)$-barrels.

<u>5.2 Lemma</u>

Let E be an α-WCG locally convex space and let C be a subset of E' whose cardinality is at most α and whose $\sigma(E', E)$-closed absolutely convex envelope is $\sigma(E', E)$-compact. Then C° is a $G(\alpha)$-barrel.

<u>Proof</u>

Let A be the $\sigma(E', E)$-closed absolutely convex envelope of C , let H be the linear span of A and put $B = C^\circ (= A^\circ)$. We have $H^\circ = \bigcap_{\lambda>0}\lambda B$ and the dual of $E/\bigcap_{\lambda>0}\lambda B$ under the norm defined by B is H . Since the norm topology is coarser than the quotient topology defined by $\tau(E, E')$ it follows that $E/\bigcap_{\lambda>0}\lambda B$ is α-WCG for this norm topology. Let $\{A_i : i \in I ; \mathrm{card}(I) = \alpha\}$ be a family of $\sigma(E/\bigcap_{\lambda>0}\lambda B, H)$-compact absolutely convex sets whose union is $\sigma(E/\bigcap_{\lambda>0}\lambda B, H)$-dense in $E/\bigcap_{\lambda>0}\lambda B$.

The set L of rational or complex rational linear combinations of elements of C is a $\sigma(H, E/\bigcap_{\lambda>0}\lambda B)$-dense subset of H and card(L) $\leq \alpha \aleph_0 = \alpha$. The topology induced on each A_i by $\sigma(E/\bigcap_{\lambda>0}\lambda B, H)$ must therefore have a base of at most α sets since it coincides with the coarsest topology making the restriction to A_i of each element of L continuous. Thus each A_i has a $\sigma(E/\bigcap_{\lambda>0}\lambda B, H)$-dense subset D_i with card$(D_i) \leq \alpha$. It follows that $D = \bigcup\{D_i : i \in I\}$ is a $\sigma(E/\bigcap_{\lambda>0}\lambda B, H)$-dense subset of $E/\bigcap_{\lambda>0}\lambda B$ and card$(D) \leq \alpha$. Finally we note that the set of rational or complex rational linear combinations of the elements of D is a norm dense subset of $E/\bigcap_{\lambda>0}\lambda B$ with cardinality at most α . Δ

CHAPTER II

B-COMPLETENESS OF $E \times \phi$

§1. Introduction

In [10], page 231, van Dulst showed that a sufficient condition for barrelledness of countable codimensional subspaces of barrelled spaces is B_r-completeness of $E \times \phi$ for an arbitrary Banach space E. Although S. Saxon and M. Levin in [40] and M. Valdivia in [45], without using van Dulst's condition, showed that a countable codimensional subspace of a barrelled space is barrelled, this was probably the starting point of looking at $E \times \phi$ in this way.

Later van Dulst in [9], Theorem 1 showed that $E \times \phi$ is B-complete (B_r-complete) whenever E is a barrelled B-complete (B_r-complete) space. Subsequently Šavgulidze ([39], Theorem 2) and Smoljanov ([42], Theorem 1) showed that the product of a hypercomplete locally convex space E and ϕ is hypercomplete. These authors note that their proofs adapt with minor modifications to the B-complete case, namely, the product of a B-complete locally convex space and ϕ is B-complete. Therefore the barrelledness condition in van Dulst's Theorem is superfluous.

In Section 2 we give a proof of the B-complete locally convex case which we believe is simpler and more illuminating. Both Šavgulidze and Smoljanov use duality theory while we do not and consequently we are able to consider some non-locally convex cases. In Section 3 we get a similar result for semiconvex spaces. Later (Chapter V) we shall extend our results to locally convex algebras.

§2. B-completeness of $E \times \phi$ when E is locally convex

Let E and F be locally convex spaces and t a linear mapping of $E \times \phi$ onto F. We identify E and ϕ with the subspaces $E \times \{0\}$ and $\{0\} \times \phi$ of $E \times \phi$ in the obvious way. Put $G = t(E)$ and let H be any algebraic supplement of G in F.

Recall that if E and F are topological vector spaces, then a linear mapping t of E onto F is said to be nearly open if the closure of $t(U)$ is a neighbourhood of 0 in F for each neighbourhood U of 0 in E.

2.1 Lemma

If t is nearly open, its restriction to E is a nearly open linear mapping of E onto G.

Proof

The dimension of H is at most countable. For notational simplicity we shall only give the proof in the infinitely countable case noting that this proof may be adapted to the finite case by means of simple notational changes.

Suppose then that $e_1, e_2, e_3, \ldots$ is a basis in H, put $J(r_n) = \{\lambda e_n : |\lambda| \leq r_n\}$ where r_n is a positive number and let G_n be the linear span of $G \cup \{e_1, e_2, \ldots, e_n\}$ $(n \in \mathbb{N})$. Let U be an absolutely convex neighbourhood of 0 in E and put $B_0 = (\text{cl } t(U)) \cap G$. We have to show that B_0 is a neighbourhood of 0 in G. Clearly B_0 is a barrel in G. For each $n \in \mathbb{N}$ we define inductively

$$B_n = \begin{cases} B_{n-1} + J(r_n) & \text{if } B_{n-1} \text{ is closed in } G_n, \\ (\text{cl } B_{n-1}) \cap G_n & \text{otherwise.} \end{cases}$$

For each $n \in \mathbb{N}$, B_n is absolutely convex. Suppose B_{n-1} is absorbent in G_{n-1}. Then in either case B_n is an absolutely convex set which spans G_n. Thus B_n is absorbent in G_n. If B_{n-1} is closed in G_n then $B_{n-1} + J(r_n)$ is closed in G_n, $J(r_n)$ being compact. It follows by induction that B_n is a barrel in G_n for each $n \in \mathbb{N}$.

We have $B_{n-1} \subseteq B_n$, $B_n \cap G = B_0$ $(n \in \mathbb{N})$ and $\bigcup_{n=1}^{\infty} B_n$ is an absolutely convex absorbent subset of F. Further $t^{-1}(\bigcup_{n=1}^{\infty} B_n)$ is an absolutely convex absorbent subset of $E \times \phi$ which induces neighbourhoods of 0 on E and ϕ, for it contains U and ϕ has its finest locally convex topology. Thus $t^{-1}(\bigcup_{n=1}^{\infty} B_n)$ is a neighbourhood of 0 in $E \times \phi$ since the product topology on $E \times \phi$ is the finest locally convex topology which induces the original topologies on E and on ϕ. Since t is nearly open $\mathrm{cl}(\bigcup_{n=1}^{\infty} B_n)$ is therefore a neighbourhood of 0 in F.

Now if we show that $\mathrm{cl}(\bigcup_{n=1}^{\infty} B_n) \subseteq 2 \bigcup_{n=1}^{\infty} \mathrm{cl}\, B_n$ then $\bigcup_{n=1}^{\infty} \mathrm{cl}\, B_n$ would be a neighbourhood of 0 in F and since

$$G \cap (\bigcup_{n=1}^{\infty} \mathrm{cl}\, B_n) = \bigcup_{n=1}^{\infty} (G \cap \mathrm{cl}\, B_n) = \bigcup_{n=1}^{\infty} (G \cap B_n) = \bigcup_{n=1}^{\infty} B_0 = B_0 ,$$

we would have that B_0 is a neighbourhood of 0 in G, as required.

Therefore suppose $x \notin \bigcup_{n=1}^{\infty} \mathrm{cl}\, B_n$. For each $n \in \mathbb{N}$ we can find an absolutely convex neighbourhood of 0 in F, say V_n, such that $x \notin B_n + V_n$. Put $W = \bigcap_{n=1}^{\infty} \mathrm{cl}(B_n + V_n)$. W is clearly a closed absolutely convex subset of F. It is also absorbent in F, for, let $y \in F$ and choose $m \in \mathbb{N}$ such that $y \in G_m$. Since B_m is absorbent in G_m, there exists $\lambda > 0$

such that $y \in \lambda B_m \subseteq \lambda\, cl(B_n + V_n)\ \forall n \geq m$. Also since V_n is absorbent in F we can find $\mu > 0$ such that $y \in \mu\, cl(B_n + V_n)$, $(n = 1, 2, \ldots, m)$. Thus for $\delta = \max\{\lambda, \mu\}$ we have $y \in \delta W$. Hence, as before, $t^{-1}(W)$ is a neighbourhood of 0 in $E \times \phi$ so that W (= cl W) is a neighbourhood of 0 in F . But

$$\bigcup_{n=1}^{\infty} B_n + W \subseteq \bigcup_{n=1}^{\infty} (B_n + B_n + V_n + V_n) = 2 \bigcup_{n=1}^{\infty} (B_n + V_n) .$$

So we have $2x \notin \bigcup_{n=1}^{\infty} B_n + W$ which implies that $2x \notin cl(\bigcup_{n=1}^{\infty} B_n)$. Δ

2.2 Lemma

Suppose that t is nearly open and continuous and G is closed in F . Then F induces the finest locally convex topology on H and F is the topological direct sum of G and H with their induced topologies.

Proof

Let ξ denote the topology of F and η the finest locally convex topology on H . The direct sum topology on F defined by $G[\xi|_G]$ and $H[\eta]$ is finer than ξ . So all we need is to show that ξ is finer than the direct sum topology of $G[\xi|_G] \oplus H[\eta]$.

The Lemma is standard if the dimension of H is finite, (see e.g. [13], Ch. I, Part 12, Theorem 7, Corollary 3). Otherwise let $e_1, e_2, e_3, \ldots$ be a basis in H . A base of neighbourhoods of 0 for the topology η on H is given by all sets of the form

$$\bigcup_{m=1}^{\infty} \sum_{n=1}^{m} J(r_n) ,$$

where $J(r_n)$ is defined as in the proof of Lemma 2.1. Let $\mathcal{V}$ be a base of closed absolutely convex ξ-neighbourhoods of 0 .

A base of neighbourhoods of 0 for the direct sum topology on F defined by $\xi|_G$ and η is given by all sets of the form

$$B = V \cap G + \bigcup_{m=1}^{\infty} \sum_{n=1}^{m} J(r_n)$$

where $V \in \mathcal{V}$. We have to show that each such B is a ξ-neighbourhood of 0 in F.

Since t is continuous, $U = t|_E^{-1}(V \cap G)$ is a closed absolutely convex neighbourhood of 0 in E. Take this U for the U in the proof of Lemma 2.1. Since V and G are closed in F we then have $B_0 = V \cap G$ which is closed in F and hence in G. Therefore, for each $m \in \mathbb{N}$

$$B_m = V \cap G + \sum_{n=1}^{m} J(r_n)$$

is closed in F. Consequently

$$\bigcup_{m=1}^{\infty} B_m = V \cap G + \bigcup_{m=1}^{\infty} \sum_{n=1}^{m} J(r_n) = B$$

is a ξ-neighbourhood of 0 in F. Δ

The notion of B-completeness is due to V. Pták [34]. It plays an important role in generalizing the closed graph and open mapping theorems. Among various references see e.g. [37], [41] and [16].

2.3 Definitions

Let E be a locally convex space. A subset A' of E' is called <u>nearly closed</u> if $A' \cap U^\circ$ is $\sigma(E', E)$-closed for every neighbourhood U of 0 in E. Then E is called <u>B-complete</u> <u>(B_r-complete)</u> if every nearly closed subspace (dense subspace) of E' is $\sigma(E', E)$-closed (is identical with E') .

We have ([41], Ch. IV, Theorem 8.3):

E is B-complete if and only if every continuous nearly open linear mapping of E onto any locally convex space F is open.

E is B_r-complete if and only if every continuous, one-to-one, nearly open linear mapping of E onto any locally convex space F is open.

Obviously the class of B-complete spaces is a subclass of the class of B_r-complete spaces, but it is not known to us if they are the same.

<u>2.4 Theorem</u>

The product of a B-complete (B_r-complete) locally convex space and the space ϕ is B-complete (B_r-complete).

<u>Proof</u>

We deal first with the B-complete case. Let E be a B-complete locally convex space and t a continuous nearly open linear mapping of $E \times \phi$ onto a locally convex space F . $t|_E$ is then a continuous linear mapping of E onto $t(E)$ which is also nearly open by Lemma 2.1. Thus $t(E)$ is B-complete by [41], Ch. IV, Theorem 8.3, Corollary 2. Since every B-complete locally convex space is complete, ([41], Ch. IV, Theorem 8.1), $t(E)$ is complete and hence closed in F . Therefore the conditions of Lemma 2.2 hold.

Now if W is any neighbourhood of the origin in $E \times \phi$ we can find absolutely convex neighbourhoods of the origin U, V in

E, ϕ respectively such that $U \times V \subseteq W$. Let H be an algebraic supplement of $t(E)$ in F. Since $t|_E$ is open onto $t(E)$ by hypothesis, the absolutely convex absorbent subset $t(U \times V)$ of F induces neighbourhoods of 0 on $t(E)$ and on H, by Lemma 2.2. Again by Lemma 2.2 we deduce that $t(U \times V)$ and therefore $t(W)$ are neighbourhoods of 0 in F. Thus t is open which implies that $E \times \phi$ is B-complete.

If we assume further that t is one-to-one and take a B_r-complete locally convex space E, we can repeat the argument to show that $E \times \phi$ is B_r-complete. Only we need notice that the image of a B_r-complete locally convex space by a continuous nearly open one-to-one linear mapping is B_r-complete and every B_r-complete locally convex space is complete, by [41], Chapter IV, Theorem 8.1. Δ

Remark

As we see from above, in order to use the argument of Lemma 2.2 for the B_r-complete case we only need t to be a continuous nearly open linear mapping such that $t|_E$ is one-to-one.

The following proposition suggests another way of looking at the problem. It is possible to use this proposition along with Lemma 2.2 to give a proof for Theorem 2.4. Although we are not going to do that we are presenting the proposition here for its own interest.

2.5 Proposition

Let E, F be locally convex spaces and t a nearly open linear mapping of $E \times \phi$ onto F. Let U be an absolutely convex neighbourhood of 0 in E, put $C = cl[t(U \times \{0\})]$ and let G

be the linear span of C . Then G is closed in F .

<u>Proof</u>

G has at most countable codimension in F . Suppose the codimension is infinite and let $e_1, e_2, e_3, \ldots$ be a basis for an algebraic supplement H of G in F . The proof for the finite case just requires notational changes. For each $r \in \mathbb{N}$ choose $\delta_r > 0$ and let $J(\delta_r) = \{\lambda e_r : |\lambda| \leq \delta_r\}$. Let $x_0 \in \text{cl}\, G$ and let $\mathcal{G}$ be any filter in G which converges to x_0 . Let $\mathcal{V}$ be a base of absolutely convex neighbourhoods of 0 in F and let $\mathcal{F}$ be the filter in F with base $\{Y + V : Y \in \mathcal{G}, V \in \mathcal{V}\}$. $\mathcal{F}$ also converges to x_0 .

For some $n \in \mathbb{N}$, $\mathcal{F}$ induces a filter on

$$D_n = nC + \sum_{r=1}^{n} J(\delta_r) .$$

If not, then there must be sequences (Y_n) in $\mathcal{G}$ and (V_n) in $\mathcal{V}$ such that for each $n \in \mathbb{N}$

$$(Y_n + V_n) \cap D_n = \emptyset \quad \text{and} \quad 2V_{n+1} \subseteq V_n . \tag{1}$$

Let

$$W_n = \text{cl}\{\tfrac{1}{2}nC + \sum_{r=1}^{n} (V_{r+1} \cap \tfrac{1}{2}J(\delta_r)) + V_{n+1}\} .$$

Each W_n is a closed absolutely convex neighbourhood of 0 in F .

Consider $W = \bigcap_{n=1}^{\infty} W_n$. We want to show that W is a neighbourhood of 0 in F . We have

$$A = \tfrac{1}{2}C + \bigcup_{n=1}^{\infty} \sum_{r=1}^{n} (V_{r+1} \cap \tfrac{1}{2}J(\delta_r)) \subseteq W .$$

For, if $m \le n$

$$\sum_{r=1}^{m} (V_{r+1} \cap \tfrac{1}{2}J(\delta_r)) \subseteq \sum_{r=1}^{n} (V_{r+1} \cap \tfrac{1}{2}J(\delta_r))$$

and if $m > n$, by using (1),

$$\sum_{r=1}^{m} (V_{r+1} \cap \tfrac{1}{2}J(\delta_r)) \subseteq \sum_{r=1}^{n} (V_{r+1} \cap \tfrac{1}{2}J(\delta_r)) + \sum_{r=n+1}^{m} V_{r+1}$$

$$\subseteq \sum_{r=1}^{n} (V_{r+1} \cap \tfrac{1}{2}J(\delta_r)) + V_{n+1} .$$

Therefore, for each $n \in \mathbb{N}$,

$$A \subseteq \tfrac{1}{2}C + \sum_{r=1}^{n} (V_{r+1} \cap \tfrac{1}{2}J(\delta_r)) + V_{n+1} \subseteq W_n .$$

Hence $A \subseteq \bigcap_{n=1}^{\infty} W_n = W$. Since W is closed we have also $\mathrm{cl}\, A \subseteq W$. Now since A is absolutely convex and absorbent in F, $t^{-1}(\mathrm{cl}\, A)$ is absolutely convex and absorbent in $E \times \phi$. Further $t^{-1}(\mathrm{cl}\, A)$ induces neighbourhoods of 0 on $E \times \{0\}$ and on $\{0\} \times \phi$, since $\tfrac{1}{2}U \times \{0\} \subseteq t^{-1}(\mathrm{cl}\, A)$ and ϕ has its finest locally convex topology. Consequently $t^{-1}(\mathrm{cl}\, A)$ is a neighbourhood of 0 in $E \times \phi$ and since t is nearly open, it follows that $\mathrm{cl}\, A$ and hence W are neighbourhoods of 0 in F.

Since $\mathcal{G}$ is a Cauchy filter, for each $n \in \mathbb{N}$, we can choose $X_n \in \mathcal{G}$ such that $X_n - X_n \subseteq \tfrac{1}{2}W_n$. We now show that

$$(X_n + \tfrac{1}{2}W_n) \cap (\tfrac{1}{2}nC + \sum_{r=1}^{n} \tfrac{1}{2}J(\delta_r)) = \emptyset , \quad (n \in \mathbb{N}) . \qquad (2)$$

Let $z \in X_n + \frac{1}{2}W_n$, $y \in X_n \cap Y_n$ $(\neq \emptyset)$. Then for some $x_n \in X_n$, $w \in \frac{1}{2}W_n$ we have

$$z = x_n + w = y + (x_n - y) + w \in Y_n + \tfrac{1}{2}W_n + \tfrac{1}{2}W_n$$

$$= Y_n + W_n$$

$$\subseteq Y_n + \tfrac{1}{2}nC + \Big(\sum_{r=1}^{n} \tfrac{1}{2}J(\delta_r)\Big) + V_n \ .$$

Hence there are elements

$$y_n \in Y_n \ , \ z' \in \tfrac{1}{2}nC + \sum_{r=1}^{n} \tfrac{1}{2}J(\delta_r) \quad \text{and} \quad v_n \in V_n$$

with $z = y_n + z' + v_n$, and so $z - z' = y_n + v_n$. This implies that

$$z \notin \tfrac{1}{2}nC + \sum_{r=1}^{n} \tfrac{1}{2}J(\delta_r) \ ,$$

for, otherwise we must have $z - z' \in D_n$ while $y_n + v_n \in Y_n + V_n$ which contradicts (1).

Choose $X \in \mathfrak{G}$ such that $X - X \subseteq \frac{1}{2}W$. Suppose $x \in X$. Since C absorbs every element of G , there exists $n_0 \in \mathbb{N}$ such that $x \in \frac{1}{2}n_0C$. For any $y_0 \in X_{n_0}$ we must have $x - y_0 \notin \frac{1}{2}W_{n_0}$ by (2) . We now deduce that $y_0 \notin X$ and consequently $X \cap X_{n_0} = \emptyset$ which is impossible, since $X, X_{n_0} \in \mathfrak{G}$.

It now follows that $\mathfrak{F}$ induces a filter on D_{n_1} say, which also converges to x_0 . But D_{n_1} being the sum of a closed set and a compact set is closed and so $x_0 \in D_{n_1}$. This shows that the closure of G is contained in

$$\bigcup_{n=1}^{\infty} D_n = G + \bigcup_{n=1}^{\infty} \sum_{r=1}^{n} J(\delta_r)$$

for any choice of the (δ_r). Since the intersection of these sets is G, we deduce that G is closed. Δ

In [45], M. Valdivia used a property of Cauchy filters on locally convex spaces to establish his Lemma 1, as a result of which he showed that a countable codimensional subspace of a barrelled space is barrelled ([45], Theorem 3). The same method has been adapted in [17], Lemma 1 by T. Husain and I. Tweddle and also in [2], §16 (14) to get similar results for semiconvex and topological vector spaces respectively. In the above proposition we have modified that method where the role played by barrelledness in their proofs has been taken over by nearly openness in ours.

Analogues of Lemma 2.2 have been used by Saxon and Levin ([40], Proposition) and by Husain and Tweddle ([17], Lemma 2). In the first paper the space is a Mackey space whose dual is weak*-sequentially complete while in the second it is a countably hyper-barrelled space. In both cases some barrelledness property is being used which we again replace by the nearly open property.

§3. <u>B-completeness of $E \times \phi$ when E is not locally convex</u>

In [19], Iyahen introduced and studied some notions on semiconvex spaces similar to those on locally convex spaces. Subsequently T. Husain and I. Tweddle in [17] investigated the inheritance of properties by countable codimensional subspaces of semiconvex spaces analogous to the works of Saxon and Levin [40] and Valdivia [45] on locally convex spaces. In continuation of this effort here we obtain a version of Theorem 2.4 for semiconvex spaces.

Also note that in 3.1 we do not need to specify the type of the range space. Since if E is semiconvex so also is a continuous nearly open image of E. Eberhardt ([11], §4(1)) has pointed out this for locally convex spaces.

If we try to prove Theorem 2.4 in the class of semiconvex spaces, following the argument of Section 2, in Lemma 2.1 we have to assume U, the neighbourhood of 0 in E, to be balanced and λ-convex for some $\lambda > 0$, where without loss of generality we take $\lambda \geq 2$. All the B_n are then balanced, λ-convex and $\mathrm{cl}(\bigcup_{n=1}^{\infty} B_n)$ is a neighbourhood of 0 in F as before.

We show this time that $\mathrm{cl}(\bigcup_{n=1}^{\infty} B_n) \subseteq \lambda \bigcup_{n=1}^{\infty} \mathrm{cl}\, B_n$. Suppose $x \notin \bigcup_{n=1}^{\infty} \mathrm{cl}\, B_n$. For each $n \in \mathbb{N}$ we can find a balanced neighbourhood V_n of 0 in F such that $x \notin B_n + V_n$. We may assume

$$V_{n+1} + V_{n+1} + V_{n+1} + V_{n+1} \subseteq V_n \qquad (n \in \mathbb{N}) .$$

Let $A_0 = B_0$ and define A_n $(n \in \mathbb{N})$ inductively as follows:

$$A_n = \begin{cases} (A_{n-1} + J(r_n) \cap V_n) \cap B_n & \text{if } A_{n-1} \text{ is closed in } G_n , \\ (\mathrm{cl}\, A_{n-1}) \cap G_n & \text{otherwise.} \end{cases}$$

Each A_n is λ-convex (note that $J(r_n) \cap V_n$ is in fact convex), balanced and absorbent in G_n with $A_{n-1} \subseteq A_n \subseteq B_n$ $(n \in \mathbb{N})$. Now we want to show that $W = \bigcap_{n=1}^{\infty} \mathrm{cl}(A_n + V_n)$ is a neighbourhood of 0 in F and

$$\bigcup_{n=1}^{\infty} B_n + W \subseteq \lambda \bigcup_{n=1}^{\infty} (B_n + V_n) .$$

First, W is λ-convex, for, as $A_n \subseteq A_{n-1} + V_n$ and $\lambda \geq 2$, we have for $n \geq 2$,

$$\mathrm{cl}(A_n + V_n) + \mathrm{cl}(A_n + V_n) \subseteq \mathrm{cl}(A_n + A_n + V_n + V_n)$$

$$\subseteq \mathrm{cl}(A_{n-1} + A_{n-1} + V_n + V_n + V_n + V_n)$$

$$\subseteq \lambda\, \mathrm{cl}(A_{n-1} + V_{n-1}) .$$

Therefore

$$W + W \subseteq \bigcap_{n=2}^{\infty} \mathrm{cl}(A_n + V_n + A_n + V_n) \subseteq \lambda \bigcap_{n=2}^{\infty} \mathrm{cl}(A_{n-1} + V_{n-1}) = \lambda W .$$

As before, W is balanced and absorbent and $t^{-1}(W)$ is a neighbourhood of 0 in $E \times \phi$. Consequently W is a neighbourhood of 0 in F . Now we have

$$\frac{1}{\lambda} \bigcup_{n=1}^{\infty} B_n + \frac{1}{\lambda} W \subseteq \frac{1}{\lambda} \bigcup_{n=1}^{\infty} (B_n + \mathrm{cl}(A_{n+1} + V_{n+1}))$$

$$\subseteq \frac{1}{\lambda} \bigcup_{n=1}^{\infty} (B_n + A_{n+1} + V_{n+1} + V_{n+1})$$

$$\subseteq \frac{1}{\lambda} \bigcup_{n=1}^{\infty} (B_n + A_n + V_{n+1} + V_{n+1} + V_{n+1})$$

$$\subseteq \frac{1}{\lambda} \bigcup_{n=1}^{\infty} (B_n + B_n + V_n)$$

$$\subseteq \frac{1}{\lambda} \bigcup_{n=1}^{\infty} (\lambda B_n + V_n) \subseteq \bigcup_{n=1}^{\infty} (B_n + V_n) ,$$

which implies that $x \notin \frac{1}{\lambda} \mathrm{cl}(\bigcup_{n=1}^{\infty} B_n)$.

So the argument of Lemma 2.1 works also for semiconvex spaces. Also it is easily seen that if t is also continuous and E is B-complete (or t is one-to-one, continuous and E is B_r-complete) then $t(E)$ is B-complete (B_r-complete); therefore by [2], §10, (9) $t(E)$ would be complete hence closed.

Since the finest locally convex topology and the finest linear topology on countable dimensional vector spaces coincide, Lemma 2.2 holds for semiconvex spaces. Now the rest of the argument in almost the same way enables us to establish:

3.2 Theorem

The product of a B-complete (B_r-complete) semiconvex space and the space ϕ is again B-complete (B_r-complete).

As a result of the attempts made to find classes of spaces in the category of topological vector spaces for which well known principles of functional analysis hold true, W. Robertson [35] defined ultrabarrelled spaces. She called a topological vector space $E[\eta]$ ultrabarrelled if any topology on E, compatible with the algebraic structure of E, with a base of η-closed neighbourhoods of the origin is necessarily coarser than η ([35], Section 5).

S.O. Iyahen in [20], Theorem 3.1 characterised ultrabarrelled spaces as follows:

A topological vector space E is ultrabarrelled if and only if every ultrabarrel in E is a neighbourhood of 0; here an ultrabarrel is a closed and balanced subset B of E for which there exists a sequence (B_n) of balanced absorbent subsets of E such that $B_1 + B_1 \subseteq B$ and $B_{n+1} + B_{n+1} \subseteq B_n$ for all $n \in \mathbb{N}$ ([20], Definition 3.1).

Alternatively in [2], §1 the authors called a sequence $\mathcal{U} = (U_n)$ of subsets U_n of a vector space E a string in E if every $U_n \in \mathcal{U}$ is balanced and absorbent and $U_{n+1} + U_{n+1} \subseteq U_n$

for all $n \in \mathbb{N}$. If E is a topological vector space then $\mathcal{U}$ is called a <u>topological string</u> if U_n is a neighbourhood of 0 $(n \in \mathbb{N})$. A topological vector space $E[\eta]$ is called <u>barrelled in $\mathcal{L}$</u> if all closed strings (the strings $\mathcal{U} = (U_n)$ such that every U_n is closed) in $E[\eta]$ are topological.

Obviously the class of ultrabarrelled spaces coincides with the class of barrelled spaces in $\mathcal{L}$.

Now we consider the problem of extending Theorem 2.4 to the class of all topological vector spaces. For Lemma 2.1 we start with a neighbourhood U of 0 in E and a topological string $\mathcal{U} = (U_n)$ in E such that $U_1 \subseteq U$. Then if $V_n = (\mathrm{cl}\ t(U_n)) \cap G$ $(n \in \mathbb{N})$ we have that (V_n) is a closed string in G and we want to show that (V_n) is topological. We now have to "extend" the sequence of sets (V_n) instead of the single set B_0. Let us suppose therefore that we can find a sequence of closed strings $(V_{n,m})$ in G_m $(m \in \mathbb{N})$ such that

$$V_n = V_{n,1} \cap G\,,\quad V_{n,m} = V_{n,m+1} \cap G_m \quad (n, m \in \mathbb{N})\,. \tag{*}$$

Then $(t^{-1}(\bigcup_{m=1}^{\infty} V_{n,m}))$ is a topological string in $E \times \phi$ since each set is balanced and absorbent, $U_n \subseteq t^{-1}(\bigcup_{m=1}^{\infty} V_{n,m})$ $(n \in \mathbb{N})$ and ϕ has its finest linear topology. The fact that t is nearly open then gives us that $(\mathrm{cl}(\bigcup_{m=1}^{\infty} V_{n,m}))$ is a topological string in F. Following [2], pp.90-91, we get

$$\mathrm{cl}(\bigcup_{m=1}^{\infty} V_{n,m}) \subseteq \bigcup_{m=1}^{\infty} \mathrm{cl}\ V_{n,m}\,,$$

which leads to what we want, since

$$V_n = (\bigcup_{m=1}^{\infty} \mathrm{cl}\ V_{n,m}) \cap G \qquad (n \in \mathbb{N})\,.$$

Lemma 2.2 and the other parts of the argument can be easily adapted to the general topological vector space case. So the problem is to justify (*). This we have not been able to do in general. It is in fact claimed in [2], p.91, that (*) is always possible, but the authors have agreed in correspondence that there is a gap in their proof which they have been unable to resolve.

CHAPTER III

THE CLOSED GRAPH THEOREM AND COUNTABLE CODIMENSIONAL SUBSPACES

§1. Introduction

M. De Wilde in [6], Corollary 1 and S.O. Iyahen in [18], Proposition 2.1 showed that: given a topological vector space F , if the topological vector space E is such that any linear map with closed graph from E into F is continuous, then the same property holds for the finite codimensional subspaces of E . We do not know so far if a similar result holds for countable codimensional subspaces of topological vector spaces.

In Theorem 2.2 we give a variant of this idea for countable codimensional subspaces replacing E and F with some classes of topological vector spaces. This general result has been used to get several known and new results throughout Section 2. In Section 3 we consider the application of Theorem 2.2 in the setting of De Wilde's webbed spaces and also we give a Mackey closed graph theorem for countable codimensional subspaces of ultrabornological spaces.

§2. The classes $\mathcal{D}$ and $\mathcal{R}$

We begin with a lemma which is a variant of the Theorem of [6] for the countably infinite codimensional case.

2.1 Lemma

Let E, F be topological vector spaces, G a countable codimensional subspace of E and $t : G \to F$ a linear mapping whose graph is closed in $G \times F$. There exists a linear mapping $T : E \to F \times \phi$

whose graph is closed in $E \times (F \times \phi)$ such that $T(x) = (t(x), 0)$ for all $x \in G$.

Proof

By Lemma 3.2 of Chapter I it is sufficient to consider the case where the graph of t is closed in $E \times F$. Note that the codimension of the new domain obtained via Lemma 3.2 may be finite. However it is easy to see how the proof may be adapted in this case; alternatively we may use De Wilde's result. Let $e_1, e_2, e_3, \ldots$ be a basis for an algebraic supplement of G in E . Now since each $x \in E$ can be uniquely written in the form

$$x = y + \sum_{n=1}^{\infty} \lambda_n e_n \quad ,$$

where $y \in G$ and only finitely many scalars λ_n are non-zero, we may define a linear mapping $T : E \to F \times \phi$ by putting $T(x) = (t(y), (\lambda_n))$. Clearly $T(x) = (t(x), 0)$ if $x \in G$. It remains to show that the graph of T is closed in $E \times (F \times \phi)$.

Suppose

$$(x_i, (t(y_i), (\lambda_n^{(i)}))) \to (x, (z, (\lambda_n))) \quad \text{in} \quad E \times (F \times \phi)$$

where

$$x_i = y_i + \sum_{n=1}^{\infty} \lambda_n^{(i)} e_n \qquad (i \in I)$$

as above. Then $x_i \to x$ in E, $t(y_i) \to z$ in F and $(\lambda_n^{(i)}) \to (\lambda_n)$ in ϕ . Since the mapping

$$(\xi_n) \longmapsto \sum_{n=1}^{\infty} \xi_n e_n$$

of ϕ into E is continuous (ϕ has its finest linear topology) we deduce that

$$\sum_{n=1}^{\infty} \lambda_n^{(i)} e_n \to \sum_{n=1}^{\infty} \lambda_n e_n$$

in E and consequently

$$y_i = x_i - \sum_{n=1}^{\infty} \lambda_n^{(i)} e_n \to x - \sum_{n=1}^{\infty} \lambda_n e_n \quad \text{in} \quad E .$$

Thus

$$(y_i, t(y_i)) \to (x - \sum_{n=1}^{\infty} \lambda_n e_n, z) \quad \text{in} \quad E \times F$$

from which it follows that

$$x - \sum_{n=1}^{\infty} \lambda_n e_n \in G \quad \text{and} \quad t(x - \sum_{n=1}^{\infty} \lambda_n e_n) = z ,$$

since the graph of t is closed in $E \times F$. Finally

$$T(x) = T(x - \sum_{n=1}^{\infty} \lambda_n e_n + \sum_{n=1}^{\infty} \lambda_n e_n)$$

$$= (t(x - \sum_{n=1}^{\infty} \lambda_n e_n), (\lambda_n)) = (z, (\lambda_n)) . \qquad \Delta$$

Let $\mathcal{R}$ be any non-empty class of topological vector spaces with the following property:

$$F \in \mathcal{R} \implies F \times \phi \in \mathcal{R} .$$

We then define $\mathcal{D}$ to be the class of all topological vector spaces E which satisfy the following closed graph theorem:

if $F \in \mathcal{R}$ and $t : E \to F$ is a linear mapping whose graph is closed in $E \times F$ then t is continuous.

Now we can state:

2.2 Theorem

$\mathcal{D}$ contains each countable codimensional subspace of each of its elements.

Proof

Let $E \in \mathcal{D}$ and G be a countable codimensional subspace of E. For every $F \in \mathcal{R}$ and every linear mapping $t : G \to F$ with closed graph in $G \times F$, by Lemma 2.1, there is a linear mapping $T : E \to F \times \phi$ with closed graph in $E \times (F \times \phi)$ such that $T(x) = (t(x), 0)$ for all $x \in G$. Now since $F \times \phi \in \mathcal{R}$, T and hence t are continuous which implies that $G \in \mathcal{D}$, by hypothesis. Δ

We have to point out here that if we drop the condition "$F \in \mathcal{R} \Longrightarrow F \times \phi \in \mathcal{R}$" from $\mathcal{R}$ then the elements of $\mathcal{D}$ will not serve in general as a domain space for a closed graph theorem when the range space is $F \times \phi$ for some $F \in \mathcal{R}$. To see this, let $\mathcal{R}_0$ be the class of all minimal locally convex spaces (see [41], Ch. IV, Exercise 6 for definition) and let $\mathcal{D}_0$ be the class of all locally convex spaces. If $E \in \mathcal{D}_0$, $F \in \mathcal{R}_0$ and $t : E \to F$ is a linear mapping with closed graph, we show that t is continuous. Let t' be the transpose of t mapping F' into E^*. Then $t'^{-1}(E')$ is $\sigma(F', F)$-dense in F'. If $t'^{-1}(E') \neq F'$, then $\sigma(F, t'^{-1}(E'))$ would be a locally convex topology on F strictly coarser than $\sigma(F, F')$ which is impossible. Therefore $t'^{-1}(E') = F'$ and hence t is weakly continuous. But since F has the topology $\sigma(F, F')$, t is in fact continuous. Now if we take for example $E = \phi[\sigma(\phi, \omega)]$ and F an arbitrary element of $\mathcal{R}_0$, the mapping $t : E \to F \times \phi$ defined by $t(\lambda_n) = (0, (\lambda_n))$ for all $(\lambda_n) \in \phi$ is a linear mapping with closed graph, because it is weakly continuous. But since ϕ (in $F \times \phi$) has its Mackey topology and $\sigma(\phi, \omega) \neq \tau(\phi, \omega)$, t is certainly not continous. Note however that $\mathcal{D}_0$ is closed under the formation of countable codimensional subspaces.

We now consider several applications of Theorem 2.2.

By Theorem 2.4, Ch. II we may take $\mathcal{R}$ to be the class of all real or complex B-complete locally convex spaces. As is well-known, the locally convex members of the corresponding class $\mathcal{D}$ are the barrelled spaces over the same scalar field. Thus we have:

2.3 Corollary ([40], Main Theorem and [45] Theorem 3)

A countable codimensional subspace of a barrelled space is barrelled.

Remark

There are of course various choices for $\mathcal{R}$ which will determine the barrelled spaces in this way. In particular we could have taken the barrelled B-complete spaces, to which van Dulst's result ([9], Theorem 1) applies. Another possibility is Valdivia's class of Γ_r-spaces.

A locally convex space E is called a Γ_r-space ([47], Definition 1) if given any subspace G of $E^*[\sigma(E^*, E)]$ such that:

(i) each subset of G which is closed and bounded is compact;

(ii) $G \cap E'$ is dense in $E'[\sigma(E', E)]$;

then we have $E' \subseteq G$.

By [47], Theorem 1, a Γ_r-space can serve as a range space for a closed graph theorem when the domain space is barrelled. Note 2 and Theorem 10 of [47] show that each B_r-complete locally convex space is a Γ_r-space and $F \times \phi$ is a Γ_r-space if F is so. Therefore $\mathcal{R}$ could be taken as the class of all Γ_r-spaces where the locally convex members of the corresponding class $\mathcal{D}$ would be the class of barrelled spaces from which we get Corollary 2.3. However we note that both van Dulst and Valdivia make use of Corollary 2.3 in establishing their results.

Let F be a B-complete space of density character at most α (α an infinite cardinal number). Suppose D_1 is a subset of F such that $\text{card}(D_1) \leq \alpha$ and D_1 is dense in F. Since ϕ is separable there is a subset D_2 of ϕ such that $\text{card}(D_2) = \aleph_0$ and D_2 is dense in ϕ. Now $\text{card}(D_1 \times D_2) \leq \alpha \aleph_0 = \alpha$ and $D_1 \times D_2$ is dense in $F \times \phi$. Hence $F \times \phi$ is a B-complete space of density character at most α. Thus we may take for $\mathcal{R}$ the class of all real or complex B-complete spaces with density character at most α. From the Remark after Theorem 5.1 of Chapter I it follows that the locally convex members of the corresponding class $\mathcal{D}$ form the class of $G(\alpha)$-barrelled spaces. Thus we have:

<u>2.4 Corollary ([32], Theorem 4)</u>

A countable codimensional subspace of a $G(\alpha)$-barrelled space is $G(\alpha)$-barrelled.

Of special interest is the case $\alpha = \aleph_0$. N. J. Kalton in [23] denoted by $\mathcal{C}(\zeta)$ the class of all locally convex spaces which serve as a domain space for the closed graph theorem when the range space is a separable B_r-complete space. Popoola and Tweddle in [32], Remarks 2 after Corollary of Theorem 1 pointed out that the $G(\mathcal{H}_0)$-barrelled spaces are the elements of $\mathfrak{G}(\zeta)$. Now let E be a locally convex space whose dual E' is $\sigma(E', E)$-sequentially complete (property (S) of [40]). By [23], Theorem 2.4 we have that $E[\tau(E, E')] \in \mathfrak{G}(\zeta)$ and so by Corollary 2.4, any countable codimensional subspace G of E under $\tau(E, E')|_G$ is in $\mathfrak{G}(\zeta)$. We want to show that G' is $\sigma(G', G)$-sequentially complete. Let (x_n) be a $\sigma(G', G)$-Cauchy sequence in G' . The closed absolutely convex envelope of (x_n) is equicontinuous by [23], Theorem 1.4, Corollary and [23], Theorem 2.6, (i) $\Rightarrow$ (iv). Hence (x_n) is convergent in $G'[\sigma(G', G)]$.

This establishes:

<u>2.5 Corollary ([27], §4, Theorem)</u>

If a locally convex space has a weak*-sequentially complete dual, then the same is true of each countable codimensional subspace.

Kalton [23] also considers the class $\mathfrak{G}(c_0)$ of all locally convex spaces which can serve as domain spaces for a closed graph theorem for linear mappings into the Banach space c_0 . Suppose that $E \in \mathfrak{G}(c_0)$ and let $t : E \to c_0 \times \phi$ be a linear mapping with closed graph. Since $E[\tau(E, E')] \in \mathfrak{G}(\zeta)$ by [23], Theorem 2.4, we certainly have that t is continuous when E has this Mackey topology and hence t is weakly continuous.

To show that t is continuous for the original topology on E , say ξ , let V be a closed absolutely convex neighbourhood of 0 in $c_0 \times \phi$. Since $t^{-1}(V) = t^{-1}(V^{\circ\circ}) = (t'(V^\circ))^\circ$, where t' is the transpose of t , we have to show that $t'(V^\circ)$ is ξ-equicontinuous. But the dual of ϕ is ω , the product of countably many copies of the scalar field, and the equicontinuous subsets of ω are the sets $\prod_{n=1}^{\infty} J(r_n)$, where $J(r_n) = \{\lambda : |\lambda| \le r_n\}$ for some $r_n \ge 0$ $(n \in \mathbb{N})$, together with their subsets. Hence $V°$ is contained in some multiple of $B \oplus \prod_{n=1}^{\infty} J(r_n)$, where B is the closed unit ball of $\ell_1 = c_0'$ and $\prod_{n=1}^{\infty} J(r_n)$ is as above for suitable (r_n) . Therefore it is enough to show that $t'(B \oplus \prod_{n=1}^{\infty} J(r_n))$ is ξ-equicontinuous for all $J(r_n)$.

We note that

$$B \oplus \prod_{n=1}^{\infty} J(r_n) = B \oplus \{0\} + \{0\} \oplus \prod_{n=1}^{\infty} J(r_n)$$

while both $B \oplus \{0\}$ and $\{0\} \oplus \prod_{n=1}^{\infty} J(r_n)$ are contained in the closed absolutely convex envelopes of sequences which are weak*-convergent to the origin - in the first case use the extremal points $(e_n, 0)$ where $e_n = (\delta_{mn}) \in \ell_1$ $(n \in \mathbb{N})$ ([252, §25, 1-2); in the second case use [37], p.134, Corollary, noting that ω is metrizable under $\sigma(\omega, \phi)$. Since t' is weak*-continuous, $t'(B \oplus \{0\})$ and $t'(\{0\} \oplus \prod_{n=1}^{\infty} J(r_n))$ are both contained in the $\sigma(E', E)$-closed absolutely convex envelopes of sequences convergent to 0 under $\sigma(E', E)$. Thus these sets are ξ-equicontinuous by [23], Theorem 3.1. The required result now follows from the fact that the sum of two equicontinuous sets is equicontinuous.

Since $\phi \times \phi$ is topologically isomorphic to ϕ, it is clear that the smallest class containing the real or complex space c_0 and satisfying our requirement for $\mathcal{R}$ is just $\{c_0, c_0 \times \phi\}$ By the above the locally convex members of the corresponding class $\mathcal{D}$ are just the members of the Kalton's class $\mathfrak{S}(c_0)$.

Thus by [23], Theorem 3.1 we have:

2.6 Corollary

If a locally convex space has the property that each weak*-Cauchy sequence in its dual space is equicontinuous, then the same is true of each countable codimensional subspace.

M. Valdivia in [49] introduced dual locally complete spaces and showed that such a space could be used as a domain space for a closed graph theorem when the range space is a Λ_r-space with its weak topology.

We start with some definitions.

2.7 Definitions

Let E be a locally convex space. If B is a bounded absolutely convex subset of E , then E_B denotes the normed space over the linear span of B , with the norm defined by B .

We say that E is locally complete if for any bounded closed absolutely convex subset B of E , E_B is a Banach space.

E is called dual locally complete if $E'[\sigma(E', E)]$ is locally complete.

Now we prove a Lemma:

2.8 Lemma

Let E be a dual locally complete space and t a linear mapping of E into $\ell_2 \times \phi$ with closed graph in $E \times (\ell_2 \times \phi)$. Then t is weakly continuous.

Proof

Denote by t' the transpose of t which maps $(\ell_2 \times \phi)' = \ell_2 \oplus \omega$ into E* and t* the transpose of t from

$(\ell_2 \times \phi)^* = \ell_2^* \oplus \omega$ into E^*. We need to show that $t'(\ell_2 \oplus \omega) \subseteq E'$. Since $t'^{-1}(E')$ is dense in $\ell_2 \oplus \omega$ [under any topology of the dual pair $(\ell_2 \oplus \omega, \ell_2 \times \phi)$] and $\ell_2 \oplus \omega$ is metrizable under $\tau \equiv \tau(\ell_2 \oplus \omega, \ell_2 \times \phi)$ it is sufficient to show that the limit of any convergent sequence (y'_n) in $t'^{-1}(E')$ in the τ-topology belongs to $t'^{-1}(E')$.

By Köthe [25], §28, 3.(1), c) there is a bounded closed absolutely convex subset B of $\ell_2 \oplus \omega$ such that (y'_n) converges to y' in $(\ell_2 \oplus \omega)_B$. B is also bounded in $\ell_2^* \oplus \omega$ under $\sigma(\ell^*_2 \oplus \omega, \ell_2 \times \phi)$. Hence $t^*(B)$ is bounded in $E^*[\sigma(E^*, E)]$. On the other hand $t^*(B)$, being a continuous image of a compact set B ([37], Ch. III, Theorem 5) is $\sigma(E^*, E)$-compact and therefore $\sigma(E^*, E)$-closed. So $A = t^*(B) = t'(B)$ is a bounded, closed and absolutely convex subset of $E^*[\sigma(E^*, E)]$ and t^* is also continuous as a mapping from $(\ell_2 \times \phi)^*_B$ into E^*_A. Therefore $(t'(y'_n))_{n\in\mathbb{N}}$, which is the same as $(t^*(y'_n))_{n\in\mathbb{N}}$, converges to $t'(y')(= t^*(y'))$ both in E^*_A and under the topology $\sigma(E^*, E)$. Now $A \cap E'$ is a closed bounded absolutely convex subset of E' and $(t'(y'_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $E'_{A\cap E'}$. By hypothesis $(t'(y'_n))_{n\in\mathbb{N}}$ converges to, z' say, in $E'_{A\cap E'}$. Hence $t'(y'_n) \to z'$ in $\sigma(E', E) = \sigma(E^*, E)|_{E'}$ which implies that $t'(y') = z' \in E'$.

<u>2.9 Corollary (of Theorem 2.2)</u>

A countable codimensional subspace of a dual locally complete space is dual locally complete.

<u>Proof</u>

As before take for $\mathcal{R}$ the smallest class containing ℓ_2

and satisfying our requirement for $\mathcal{R}$, viz $\{\ell_2, \ell_2 \times \phi\}$. If $\mathcal{D}$ is the corresponding class of domain spaces and $E[\xi]$ is a dual locally complete space, then $E[\tau(E, E')] \in \mathcal{D}$ by Lemma 2.8 and [49], Theorem 4. Let G be a countable codimensional subspace of E . G , with the topology induced by $\tau(E, E')$, belongs to $\mathcal{D}$ by Theorem 2.2.

Now let t be a linear mapping with closed graph of $G[\xi|_G]$ into ℓ_2 . Since the duals of G under the topologies induced by ξ and $\tau(E, E')$ are identical, the graph of t is also closed when G has topology $\tau(E, E')|_G$. Hence t is weakly continuous. It follows from this and [49], Theorem 4 that $G[\xi|_G]$ is dual locally complete.

We now turn to a non-locally convex case.

Iyahen in [19] called a semiconvex space E <u>hyperbarrelled</u> if every closed balanced semiconvex absorbent subset of E is a neighbourhood of 0 .

From Theorem 3.2 of Chapter II we may take for $\mathcal{R}$ the class of all real or complex semiconvex B_r-complete spaces. Let E be a hyperbarrelled space and let t be a linear mapping with closed graph of E into a member F of $\mathcal{R}$. Since t is nearly continuous, by the Remark after Definition 3.1 of Chapter II, t is continuous. So E and hence by Theorem 2.2 each countable codimensional subspace of E belongs to the corresponding class of $\mathcal{D}$. On the other hand, since each complete locally bounded space belongs to our class $\mathcal{R}$, by [19], Theorem 3.3, we deduce that the semiconvex members of $\mathcal{D}$ are the hyperbarrelled spaces. Thus we have:

2.10 Corollary ([17], Theorem 6)

A countable codimensional subspace of a hyperbarrelled space is hyperbarrelled.

It is usual to define a barrelled space as a locally convex space E in which every barrel is a neighbourhood of 0. However, in view of [28], Theorem 2.2, one may look on a barrelled space as a locally convex space which serves as a domain space for a closed graph theorem where the range space is an arbitrary Banach space F.

Iyahen modified this idea by taking F as a strict inductive limit [21] or generalised strict inductive limit [22] of a sequence $(F_n)_{n\in\mathbb{N}}$ of Banach spaces. Since barrelled spaces are also called t-spaces, Iyahen called his new classes of locally convex spaces τ-spaces and T-spaces respectively. It is immediate from the definitions that every T-space is a τ-space and every τ-space is barrelled (a t-space). We can regard ϕ as a strict (generalised strict) inductive limit of the sequence $(\mathbb{K}^n)_{n\in\mathbb{N}}$ of Banach spaces. If F is a strict (generalised strict) inductive limit of the sequence $(F_n)_{n\in\mathbb{N}}$ of Banach spaces, then $F\times\phi$ would be the strict (generalised strict) inductive limit of the sequence $(F_n\times\mathbb{K}^n)_{n\in\mathbb{N}}$ of Banach spaces. Thus we establish our last corollary of Theorem 2.2 as follows:

2.11 Corollary

A countable codimensional subspace of a τ-space (resp. T-space) is a τ-space (resp. T-space).

§3. A closed graph theorem

The locally convex spaces with réseaux were defined by De Wilde in [7] and were studied further in [5]. The idea was extended to topological vector spaces by W. Robertson in [36] who used the term spaces with a web. We use here the terminology and definitions of [36] and [37]. For some generalisations of these spaces see [31]. A detailed survey of webbed spaces is available in §35 of [26]. In this Section we are adapting ideas from [36], [37] and especially from [4], Propositions IV 5.1-2 to give a closed graph theorem for countable codimensional subspaces of ultrabornological spaces and webbed spaces.

3.1 Definitions

A web $\mathcal{W}$ in a vector space E is a family

$$\{A_{n_1 n_2 \ldots n_k} : k \in \mathbb{N},\ n_1, n_2, \ldots, n_k \in \mathbb{N}\}$$

of absolutely convex subsets of E, indexed by finite sequences of positive integers, such that $\bigcup\{A_{n_1} : n_1 \in \mathbb{N}\}$ absorbs each point of E, and, for all $k \in \mathbb{N}$, $n_1, n_2, \ldots, n_k \in \mathbb{N}$,

$$A_{n_1 n_2 \ldots n_{k+1}} + A_{n_1 n_2 \ldots n_{k+1}} \subseteq A_{n_1 n_2 \ldots n_k} \text{ for each}$$

$n_{k+1} \in \mathbb{N}$,

$\bigcup\{A_{n_1 n_2 \ldots n_{k+1}} : n_{k+1} \in \mathbb{N}\}$ absorbs each point of

$A_{n_1 n_2 \ldots n_k}$.

For convenience, for each $k \in \mathbb{N}$, we shall call the sequences $(A_{n_1 n_2 \ldots n_k n_{k+1}})$, as n_{k+1} varies, the sequence determined by $A_{n_1 n_2 \ldots n_k}$.

The members of a web $\mathcal{W}$ are arranged in layers. For each $k \in \mathbb{N}$, the kth layer of $\mathcal{W}$ contains all sets of the form $A_{n_1 n_2 \dots n_k}$ where k is constant and $n_1, n_2, \dots, n_k$ are varying. To each sequence $(n_k)_{k \in \mathbb{N}}$ of positive integers corresponds a <u>strand</u> $(A_{n_1 n_2 \dots n_k})_{k \in \mathbb{N}}$, which we will denote by (W_k) when we are dealing only with one strand at a time, where $W_k = A_{n_1 n_2 \dots n_k}$, $k = 1, 2, 3, \dots$. We may also assume $W_{k+1} \subseteq \frac{1}{2} W_k$ $(k \in \mathbb{N})$, (see [37], p.156).

3.2 Definition

Todd and Saxon in [44] defined and studied some properties of unordered Baire-like spaces. They called a locally convex space E, <u>unordered Baire-like</u> if it can not be covered by any sequence of rare (nowhere dense), absolutely convex subsets. Clearly every locally convex Baire space is unordered Baire-like and every unordered Baire-like space is barrelled.

We now give a Lemma which is the analogue for a Baire-like space of Lemma 11 of [36].

3.3 Lemma

Let E be an unordered Baire-like space, let t be a linear mapping of E into a vector space F with a web $\mathcal{W}$. Then there exists a strand (W_k) of $\mathcal{W}$ such that for each $k \in \mathbb{N}$, $\mathrm{cl}\ t^{-1}(W_k)$ is a neighbourhood of 0 in E.

Proof

Without loss of generality we can suppose that the first layer of the web $\mathcal{W}$ contains the whole space F. Put $W_1 = F$. Clearly $\mathrm{cl}\ t^{-1}(W_1)$ is a neighbourhood of 0 in E and $t^{-1}(W_1) = E$ cannot be written as a union of a sequence of rare absolutely convex subsets of E. For some $k > 0$, suppose that W_k has been chosen so that $\mathrm{cl}\, t^{-1}(W_k)$ is a neighbourhood of 0 and $t^{-1}(W_k)$ cannot be written as a union of a sequence of rare absolutely convex sets. Let (A_i) be a sequence of the $(k+1)$th layer determined by W_k and for each i, let (A_{ij}) be a sequence in the $(k+2)$th layer determined by A_i. Then

$$W_k \subseteq \bigcup\{nA_i : i, n \in \mathbb{N}\} \subseteq \bigcup\{nA_{ij} : i, j, n \in \mathbb{N}\} .$$

Hence

$$t^{-1}(W_k) \subseteq \bigcup\{t^{-1}(nA_i) : i, n \in \mathbb{N}\} \text{ and for each } i ,$$

$$t^{-1}(A_i) \subseteq \bigcup\{t^{-1}(nA_{ij}) : j, n \in \mathbb{N}\} ,$$

which implies that there exist $i_0, j_0 \in \mathbb{N}$ so that $t^{-1}(A_{i_0})$ and also $t^{-1}(A_{i_0 j_0})$ cannot be written as a union of a sequence of rare absolutely convex sets. Now there exists an $x \in E$ and a neighbourhood U of 0 in E with $x + U \subseteq \mathrm{cl}\, t^{-1}(A_{i_0 j_0})$, and so

$$U = -x + (x + U) \subseteq \mathrm{cl}\, t^{-1}(A_{i_0 j_0}) + \mathrm{cl}\, t^{-1}(A_{i_0 j_0})$$

$$\subseteq \mathrm{cl}\, t^{-1}(A_{i_0 j_0} + A_{i_0 j_0}) \subseteq \mathrm{cl}\, t^{-1}(A_{i_0}) .$$

Put $W_{k+1} = A_{i_0}$, then $t^{-1}(W_{k+1})$ is a neighbourhood of 0 and $t^{-1}(W_{k+1})$ cannot be written as a union of a sequence of rare absolutely convex sets. This completes the proof by induction. Δ

<u>3.4 Definition</u>

A web $\mathcal{W}$ in a locally convex space E is said to be <u>compatible</u> with the topology if, for each strand (W_k) of $\mathcal{W}$ and each neighbourhood V of 0, there is a $k \in \mathbb{N}$ such that

$W_k \subseteq V$. This is equivalent to saying that, for each strand (W_k) of $\mathcal{W}$ and each $x_k \in W_k$ we have $x_k \to 0$.

$\mathcal{W}$ is called a completing web if, for each strand (W_k) , the series $\sum_{k=1}^{\infty} x_k$ is convergent for every choice of $x_k \in W_k$.

A completing web is compatible; the converse also holds if the space is sequentially complete ([37], Appendix, Section 1).

3.5 Definition

A sequence $(x_n)_{n \in \mathbb{N}}$ in a locally convex space E is Mackey convergent (resp. fast convergent) to x if there is a bounded absolutely convex subset B of E such that $x_n \to x$ in E_B (and E_B is complete).

Clearly a fast convergent sequence is Mackey convergent and, in a metrizable locally convex space, every convergent sequence is Mackey convergent by [25], §28, 3.

We say that a subset A of a locally convex space E is Mackey closed (fast sequentially closed) if it contains the limits of all Mackey convergent (fast convergent) sequences of E contained in A .

3.6 Lemma

Let F be a locally convex space with a completing web $\mathcal{W}$. For each strand (W_k) of $\mathcal{W}$ there exists a sequence (λ_k) of positive numbers such that $\sum_{k=1}^{\infty} \mu_k y_k$ and $(\mu_k y_k)_{k \in \mathbb{N}}$ are fast convergent for all μ_k with $|\mu_k| \leq \lambda_k$ and $y_k \in W_k$.

Proof

Since the elements of a strand are balanced, every strand of a completing web would be a completing sequence in the sense of De Wilde [4], Definition IV, 1.3. Then the result follows by [4], Proposition IV, 1.9. Δ

Recall that an ultrabornological space is a locally convex space E which is an inductive limit of a family $\{E_i[\xi_i] : i \in I\}$ of Banach spaces. It follows from [4], Proposition III, 2.2 that in fact E is the inductive limit of the family of spaces E_B where B is a bounded absolutely convex subset of E such that E_B is complete.

3.7 Theorem

Let G be a countable codimensional subspace of an ultra-bornological space E and let F be a locally convex space with a completing web. If t is a linear mapping of G into F whose graph is Mackey closed in $G \times F$ then t is continuous.

Proof

Suppose E is the inductive limit of the family $\{E_i[\xi_i] : i \in I\}$ of Banach spaces where each $E_i[\xi_i]$ is an E_B as above. Regarding each $E_i (i \in I)$ as a subspace of E, G is a subspace of $G + E_i$ of at most countable codimension. By [46], Theorem 3, G is then the inductive limit of the family of normed spaces $\{(E_i \cap G)[\xi_i] : i \in I\}$, where for each $i \in I$, $G_i[\xi_i] = (E_i \cap G)[\xi_i]$ is a countable codimensional subspace of the Banach space $E_i[\xi_i]$. If f_i is the natural embedding of G_i into G $(i \in I)$, it is clear that for each $i \in I$ the graph of $t \circ f_i$ is Mackey closed in $G_i \times F$. Consequently it is enough to establish the theorem when G itself is a countable codimensional subspace of a Banach space. Then since Banach spaces are unordered Baire-like, it follows from [44], Theorem 4.4 that G is unordered Baire-like.

Let $\mathcal{W}$ be a completing web in F. By Lemma 3.3 there is a strand (W_k) of $\mathcal{W}$ such that $\mathrm{cl}\, t^{-1}(W_k)$ is a neighbourhood of 0 in G $(k \in \mathbb{N})$. Then by Lemma 3.6 there is a sequence (λ_k) of positive numbers, where we may assume $\lambda_k \le 1$ $(k \in \mathbb{N})$ such that $\sum_{k=1}^{\infty} \lambda_k y_k$ is fast convergent $\forall\, y_k \in W_k$ $(k \in \mathbb{N})$. Now there is a decreasing sequence $(U_k)_{k\in\mathbb{N}}$ of neighbourhoods of 0 in G, which we may assume without loss of generality to be a base of neighbourhoods of 0 in G, such that $U_k \subseteq \lambda_k\, \mathrm{cl}\, t^{-1}(W_k)$ $(k \in \mathbb{N})$. Thus $U_k \subseteq \lambda_k\, t^{-1}(W_k) + U_{k+1}$ and hence $t(U_k) \subseteq t(U_{k+1}) + \lambda_k W_k$ $(k \in \mathbb{N})$.

For any closed absolutely convex neighbourhood V of 0 in F, since $\mathcal{W}$ is compatible, there is a $k_0 \in \mathbb{N}$ such that $W_{k_0-1} \subseteq V$. We show that $t(U_{k_0}) \subseteq V$. Take $x_0 \in U_{k_0}$. There exists $x_1 \in U_{k_0+1}$ such that $t(x_0) - t(x_1) \in \lambda_{k_0} W_{k_0}$. Again there exists $x_2 \in U_{k_0+2}$ such that $t(x_1) - t(x_2) \in \lambda_{k_0+1} W_{k_0+1}$, and so on. In general we choose $x_r \in U_{k_0+r}$ such that $t(x_r) - t(x_{r+1}) \in \lambda_{k_0+r} W_{k_0+r}$. By Lemma 3.6

$$\sum_{r=0}^{\infty} \lambda_{k_0+r} \frac{t(x_r) - t(x_{r+1})}{\lambda_{k_0} + r} = \sum_{r=0}^{\infty} (t(x_r) - t(x_{r+1}))$$

$$= \lim_{r\to\infty} \{t(x_0) - t(x_r)\}$$

is fast convergent. Therefore $t(x_r)$ is fast convergent and hence Mackey convergent to y say. Since $x_r \to 0$ in G, and G is metrizable, x_r is Mackey convergent to the origin. Since the graph of t is Mackey closed, we have $y = 0$. Thus $t(x_r) \to 0$ in F. Also if $r > 0$

$$t(x_0) - t(x_r) = \sum_{s<r} (t(x_s) - t(x_{s+1}))$$

$$\in W_{k_0} + W_{k_0+1} + \ldots + W_{k_0+r-1}$$

$$\subseteq \frac{1}{2} W_{k_0-1} + \frac{1}{2^2} W_{k_0-1} + \ldots + \frac{1}{2^r} W_{k_0-1} \subseteq W_{k_0-1} .$$

Hence

$$\lim_{r\to\infty}(t(x_0) - t(x_r)) = t(x_0)$$

$$\in \mathrm{cl}(W_{k_0-1}) \subseteq \mathrm{cl}\, V = V . \qquad \Delta$$

De Wilde ([4], Ch. IV) defined webbed spaces in a slightly different way. The class of all locally convex spaces with a completing web is a subclass of De Wilde's webbed spaces consisting of those for which the elements of the web are all absolutely convex. He also defined strictly webbed spaces - the locally convex spaces with a completing web $\mathcal{W}$ such that for each strand (W_k) of $\mathcal{W}$ a sequence $(\lambda_k)_{k\in\mathbb{N}}$ of positive numbers can be chosen in such a way that

$$\sum_{k=k_0+1}^{\infty} \mu_k x_k \in W_{k_0}$$

for all $k_0 \in \mathbb{N}$, $x_k \in W_k$ and $0 \le \mu_k \le \lambda_k$.

Let $\mathcal{R}_0$ be one of the following classes of locally convex spaces: (i) the class of De Wilde's webbed spaces, (ii) the class of strictly webbed spaces, (iii) the class of spaces with a completing web. By Propositions IV 4.8-9 of [4], $F^{\times}\phi$ belongs to $\mathcal{R}_0$ for every $F \in \mathcal{R}_0$. So these classes fulfil the requirement for $\mathcal{R}$ of Section 2. Since every closed subset of a locally convex space is fast sequentially closed, it follows from Propositions IV 5.1-2 of [4] that in either case an important subclass of the corresponding class $\mathcal{D}$ is formed by the real or complex ultrabornological spaces, and hence by Theorem 2.2 we get a variant of Theorem 3.7:

3.8 Proposition

Every linear mapping from a countable codimensional subspace of an ultrabornological space into a webbed space, strictly webbed space or a space with a completing web whose graph is closed is continuous.

We note that such a subspace is barrelled (Corollary 2.3) and bornological ([46], Corollary 1.3).

CHAPTER IV

M(α)-BARRELLED SPACES

§1. Introduction

Throughout this Chapter α is an infinite cardinal number.

M. Valdivia in [50] calls a locally convex space E α-barrelled if every bounded subset of $E'[\sigma(E', E)]$ whose cardinal number is not larger than α, is equicontinuous. An $\aleph_0$-barrelled space is also said to be ω-barrelled [27] or σ-barrelled [8]. Each α-barrelled space is G(α)-barrelled (Remarks 1 after Corollary of Theorem 1 of [32]). But the converse is not true in general by [32], Examples (ii)-(iv).

A. Marquina in [29] gave the following closed graph theorem for α-barrelled spaces.

1.1 Theorem ([29], Theorem 1)

Let E be a Mackey α-barrelled space. Let F be an α-WCG Banach space. If t is a linear mapping from E into F with closed graph in $E \times F$, then t is continuous.

In this chapter we are concerned with the connection between the above closed graph theorem and the one introduced in Chapter I, 5.1.

Let $\mathcal{R}_1(\alpha)$ be the class of all Banach spaces with density character at most α and let $\mathcal{R}_2(\alpha)$ be the class of all α-WCG Banach spaces. Denote by $\mathcal{D}_i(\alpha)$ the class of all locally convex spaces E with the property that, whenever $t : E \to F$ is a linear mapping with closed graph of E into an arbitrary element F of $\mathcal{R}_i(\alpha)$ then t is continuous $(i = 1, 2)$.

The class $\mathcal{D}_1(\alpha)$ consists of the $G(\alpha)$-barrelled spaces by [32], Theorem 3. By Theorem 1.1 each Mackey α-barrelled locally convex space belongs to $\mathcal{D}_2(\alpha)$. Our main purpose in this chapter is to describe the elements of $\mathcal{D}_2(\alpha)$. Since each Banach space of density character at most α is clearly α-WCG, we have $\mathcal{D}_2(\alpha) \subseteq \mathcal{D}_1(\alpha)$; conversely, in Section 2 we show that on each $G(\alpha)$-barrelled space there is a topology of the same dual pair under which it belongs to $\mathcal{D}_2(\alpha)$. Thus the same dual pairs appear in $\mathcal{D}_1(\alpha)$ and $\mathcal{D}_2(\alpha)$. For each α we give an example of an element of $\mathcal{D}_2(\alpha)$ which is neither an α-barrelled space nor a Mackey space; also we give an example of an α-barrelled space which is not in $\mathcal{D}_2(\alpha)$.

Clearly, in the definition of $\mathcal{D}_2(\alpha)$ we may replace $\mathcal{R}_2(\alpha)$ with the class of "closed subspaces of α-WCG Banach spaces". It is more convenient to work with this extended class, which we denote by $\mathcal{R}(\alpha)$. To simplify notation we shall write $\mathcal{D}(\alpha)$ in place of $\mathcal{D}_2(\alpha)$.

Note that by [15], Proposition 1.1 and page 87 of [15], the class $\mathcal{R}_2(\alpha)$ is strictly contained in $\mathcal{R}(\alpha)$ for $\alpha = \aleph_0$.

§2. G(α)-barrelled spaces and the class $\mathcal{D}(\alpha)$

The following Theorem is an extension of Marquina's closed graph theorem (Theorem 1.1).

2.1 Theorem

Let E be a $G(\alpha)$-barrelled space and let $F \in \mathcal{R}(\alpha)$. If t is a linear mapping of E into F with closed graph, then t is weakly continuous. Thus $E[\tau(E, E')] \in \mathcal{D}(\alpha)$.

Proof

If G is an α-WCG Banach space having F as a closed subspace, then we may regard t as a linear mapping of E into G and the graph of t remains closed because of the completeness of F. Thus it is enough to consider the case where F is α-WCG.

We have to show that $t'(F') \subseteq E'$. Since $t'^{-1}(E')$ is $\sigma(F', F)$-dense in F', it is sufficient to show that $t'^{-1}(E')$ is $\sigma(F', F)$-closed. If S is the closed unit ball of F', by [25], §21, 10 (6), we need only show that $X = S \cap t'^{-1}(E')$ is $\sigma(F', F)$-closed. Since S is $\sigma(F', F)$-compact and $X \subset S$, then X is $\sigma(F', F)$-closed if and only if it is $\sigma(F', F)$-compact. To show that X is $\sigma(F', F)$-compact, since F is α-WCG, by [48], Theorem 1 it is enough to show that X is $\sigma(F', F)$-α-compact.

Let $C = \{z_i : i \in I\}$, $\mathrm{card}(I) = \alpha$, be a net in X and let z_0 be a $\sigma(F', F)$-adherent point of C. z_0 certainly belongs to S. Let us show that $z_0 \in t'^{-1}(E')$ which will complete the proof. Let B be the $\sigma(F', F)$-closed absolutely convex envelope of C. Since $B \subseteq S$ and S is $\sigma(F', F)$-compact, then B is $\sigma(F', F)$-compact. Then by the Lemma 5.2 of Chapter I, $B^\circ = C^\circ$ is a $G(\alpha)$-barrel in F. So it follows from [32], Section 4, Lemma that $\mathrm{cl}(t^{-1}(C^\circ))$ is a $G(\alpha)$-barrel in E. Since $t'^{-1}(E')$ is $\sigma(F', F)$-dense in F', $(F, t'^{-1}(E'))$ is a dual pair and $t : E[\sigma(E, E')] \to F[\sigma(F, t'^{-1}(E'))]$ is obviously continuous. Hence $t' : t'^{-1}(E') \to E'$ is continous under $\sigma(t'^{-1}(E'), F)$, $\sigma(E', E)$ and from [37], Ch. II, Lemma 6 we have

$$(t'(C))^\circ = t''^{-1}(C^\circ) .$$

This implies that $t''^{-1}(C^\circ)$ is $\sigma(E, E')$-closed and since $t'' = t$ we deduce that

$$(t'(C))^\circ = t^{-1}(C^\circ) = cl(t^{-1}(C^\circ))$$

is a $G(\alpha)$-barrel and hence a neighbourhood of 0 in E. Therefore $t'(C)$ is equicontinuous and hence $t'(z_0)$ being a $\sigma(E', E)$-adherent point of $t'(C)$ belongs E', i.e. $z_0 \in t'^{-1}(E')$.

For the last part we note that closedness of the graph does not depend upon a particular topology of a dual pair and that a weakly continuous linear mapping is continuous when the domain space has its Mackey topology.

2.2 Remarks

(i) Since every Banach space of density character at most α is an α-WCG Banach space, it follows from Theorem 5.1 of Chapter I that: a locally convex space E is $G(\alpha)$-barrelled if, whenever $F \in \mathcal{R}(\alpha)$ then every linear mapping $t : E \to F$ with closed graph is continuous. In Section 3 we shall describe the coarsest locally convex topology η on the space E of Theorem 2.1 which will give the same dual space and guarantee that t is continuous whenever $F \in \mathcal{R}(\alpha)$. The first example of Section 4 shows that η can be strictly coarser than $\tau(E, E')$ for any α.

(ii) The case $\alpha = \aleph_0$ of Theorem 2.1 is Marquina's Theorem 2 in [29], (note that Marquina misses out the Mackey assumption in the Theorem but uses it in the proof). For, if E is a Mackey space such that $E'[\sigma(E', E)]$ is sequentially complete, then $E \in \mathfrak{S}(\zeta) = \mathfrak{S}(\zeta_B)$ by [23], Theorems 2.4, 2.6. But as we pointed out earlier (proof of Corollary 2.5 of Chapter III) the locally convex elements of this class of spaces are the $G(\aleph_0)$-barrelled spaces. In this connection we note that by [15], Proposition 1.1 an $\aleph_0$-WCG Banach space is in fact weakly compactly generated.

$$(t'(C))^\circ = t^{-1}(C^\circ) = cl(t^{-1}(C^\circ))$$

is a $G(\alpha)$-barrel and hence a neighbourhood of 0 in E. Therefore $t'(C)$ is equicontinuous and hence $t'(z_0)$ being a $\sigma(E', E)$-adherent point of $t'(C)$ belongs E', i.e. $z_0 \in t'^{-1}(E')$.

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§3. A characterisation of $\mathcal{D}(\alpha)$.

3.1 Definition

Let $E[\xi]$ be a locally convex space and let B be a barrel in $E[\xi]$. We shall say that B is an M(α)-barrel in $E[\xi]$ if $E/\bigcap_{\lambda>0}\lambda B$ with the norm topology defined by B is (isometrically isomorphic to) a subspace of an α-WCG Banach space.

The G(α)-barrels in $E[\xi]$ and so in particular the closed absolutely convex neighbourhoods of 0 in $E[\sigma(E, E')]$ are M(α)-barrels in $E[\xi]$. Clearly all topologies of a given dual pair determine the same M(α)-barrels.

3.2 Properties of M(α)-barrels

Let $\mathcal{M}(\alpha)$ be the set of all M(α)-barrels in a locally convex space $E[\xi]$. We have:

(i) if $B \in \mathcal{M}(\alpha)$, then $\mu B \in \mathcal{M}(\alpha)$ for any non-zero scalar μ ;

(ii) if $B_1, B_2 \in \mathcal{M}(\alpha)$, then $B_1 \cap B_2 \in \mathcal{M}(\alpha)$;

(iii) if B is a barrel in E and there exists $B_1 \in \mathcal{M}(\alpha)$ such that $B_1 \subseteq B$, then $B \in \mathcal{M}(\alpha)$.

Proof

(i) As vector spaces we have $E/\bigcap_{\lambda>0}\lambda B = E/\bigcap_{\lambda>0}\lambda(\mu B)$. Also B and μB define equivalent norms.

(ii) If P_1, P_2 are the Minkowski functionals of B_1, B_2, then $P = \max\{P_1, P_2\}$ is the Minkowski functional of $B_1 \cap B_2$. So if the product space $(E/\bigcap_{\lambda>0}\lambda B_1) \times (E/\bigcap_{\lambda>0}\lambda B_2)$ has the maximum of the norms defined by B_1, B_2 on $E/\bigcap_{\lambda>0}\lambda B_1$, $E/\bigcap_{\lambda>0}\lambda B_2$, then the product space is a subspace of an α-WCG Banach space, namely a subspace of the product of two α-WCG Banach spaces, and the mapping

$$x + \bigcap_{\lambda>0} \lambda(B_1 \cap B_2) \to (x + \bigcap_{\lambda>0} \lambda B_1, x + \bigcap_{\lambda>0}\lambda B_2)$$

from $E/\bigcap_{\lambda>0}\lambda(B_1 \cap B_2)$ onto $(E/\bigcap_{\lambda>0}\lambda B_1) \times (E/\bigcap_{\lambda>0}\lambda B_2)$ is a linear isometry. We only show that the mapping is well-defined and one-to-one, the rest is straightforward. Let

$$x + \bigcap_{\lambda>0} \lambda(B_1 \cap B_2) = y + \bigcap_{\lambda>0} \lambda(B_1 \cap B_2) .$$

Then

$$x - y \in \bigcap_{\lambda>0} \lambda(B_1 \cap B_2) = (\bigcap_{\lambda>0} \lambda B_1) \cap (\bigcap_{\lambda>0} \lambda B_2) ,$$

which implies that

$$x + \bigcap_{\lambda>0} \lambda B_1 = y + \bigcap_{\lambda>0} \lambda B_1 \quad \text{and} \quad x + \bigcap_{\lambda>0} \lambda B_2 = y + \bigcap_{\lambda>0} \lambda B_2 .$$

Now if

$$x + \bigcap_{\lambda>0} \lambda(B_1 \cap B_2) \neq y + \bigcap_{\lambda>0} \lambda(B_1 \cap B_2) ,$$

then

$$x - y \notin \bigcap_{\lambda>0} \lambda(B_1 \cap B_2) = (\bigcap_{\lambda>0} \lambda B_1) \cap (\bigcap_{\lambda>0} \lambda B_2) .$$

Hence

$$x + \bigcap_{\lambda>0} \lambda B_1 \neq y + \bigcap_{\lambda>0} \lambda B_1 \quad \text{or} \quad x + \bigcap_{\lambda>0} \lambda B_2 \neq y + \bigcap_{\lambda>0} \lambda B_2 ,$$

i.e. the mapping is one-to-one. It follows that $E/\bigcap_{\lambda>0}\lambda(B_1 \cap B_2)$ with the norm defined by $B_1 \cap B_2$ is isometrically isomorphic to a subspace of an α-WCG Banach space.

(iii) We may regard $E_1 = E/\bigcap_{\lambda>0}\lambda B_1$ with the norm topology defined by B_1 as a subspace of an α-WCG Banach space F. The dual of E_1 is the linear span of B_1° (polar in E^*) which is therefore identified with a quotient of F'. If q is the quotient map of F' onto E_1', then by the Hahn-Banach theorem B_1° is the image under q

of the closed unit ball S of F' . Since $B^\circ \subseteq B_1{}^\circ$, the set $A = q^{-1}(B^\circ) \cap S$ is an absolutely convex $\sigma(F', F)$-compact subset of S such that $q(A) = B^\circ$.

Let G be the linear span of A , let H be the linear span of B° and let s be the mapping of G onto H defined by restricting q . Then s is continuous under the topologies $\sigma(G, F/G^\circ)$ and $\sigma(H, E_1/H^\circ)$ (polar of H in E_1) . The transpose $s': E_1/H^\circ \rightarrow F/G^\circ$ is therefore continuous under $\tau(E_1/H^\circ, H)$ and $\tau(F/G^\circ, G)$ and it is also one-to-one since s is onto.

Now $\tau(F/G^\circ, G)$ is a norm topology with A as the closed unit ball of the dual space G . It is coarser than the quotient topology, so $F/G^\circ[\tau(F/G^\circ, G)]$ is α-WCG and its completion is an α-WCG Banach space. In E_1

$$x + \bigcap_{\lambda>0} \lambda B_1 \in H^\circ$$

$$\Longleftrightarrow \langle x + \bigcap_{\lambda>0} \lambda B_1, z' \rangle = 0 \quad \forall z' \in H = \bigcup_{n=1}^{\infty} nB^\circ$$

$$\Longleftrightarrow \langle x, z' \rangle = 0 \quad \forall z' \in \bigcup_{n=1}^{\infty} nB^\circ \quad (\text{since } B^\circ \subseteq B^\circ{}_1)$$

$$\Longleftrightarrow x \in (\bigcup_{n=1}^{\infty} nB^\circ)^\circ \quad (\text{polar in } E)$$

$$= \bigcap_{n=1}^{\infty} \frac{1}{n} B = \bigcap_{\lambda>0} \lambda B .$$

Thus $H^\circ = \bigcap_{\lambda>0}\lambda B/\bigcap_{\lambda>0}\lambda B_1$ and so by [25], §7.6(7), (see also remark following §15.4(4) of [25]) we have

$$E_1/H^\circ = (E/\bigcap_{\lambda>0} \lambda B_1)/(\bigcap_{\lambda>0} \lambda B/\bigcap_{\lambda>0} \lambda B_1) = E/\bigcap_{\lambda>0} \lambda B .$$

The result will follow if we show that s' is an isometry. If $x + \bigcap_{\lambda>0} \lambda B \in E/\bigcap_{\lambda>0} \lambda B$ we have

$$\begin{aligned}
\left\| x + \bigcap_{\lambda>0} \lambda B \right\| &= \sup\{ |\langle x + \bigcap_{\lambda>0} \lambda B, x' \rangle| : x' \in B^\circ \} \\
&= \sup\{ |\langle x + \bigcap_{\lambda>0} \lambda B, s(z') \rangle| : z' \in A \} \\
&= \sup\{ |\langle s'(x + \bigcap_{\lambda>0} \lambda B), z' \rangle| : z' \in A \} \\
&= \left\| s'(x + \bigcap_{\lambda>0} \lambda B) \right\| .
\end{aligned}$$

Δ

Remark

Properties (i) and (ii) show that the set of $M(\alpha)$-barrels in a locally convex space forms a base of neighbourhoods of 0 for a locally convex topology on the space (Chapter I, Theorem 3.1). By an earlier observation this topology is finer than the weak topology of the space.

3.3 Definition

We shall say that a locally convex space $E[\xi]$ is $M(\alpha)$-barrelled if each $M(\alpha)$-barrel in $E[\xi]$ is a ξ-neighbourhood of 0 .

The following theorem characterises the class $\mathcal{D}(\alpha)$.

3.4 Theorem

A locally convex space $E[\xi]$ is $M(\alpha)$-barrelled if and only if every linear mapping t of $E[\xi]$ into a closed subspace G of an α-WCG Banach space whose graph is closed is continuous.

Proof

Sufficiency: Let B be an $M(\alpha)$-barrel in E. Then the completion F of $E/\bigcap_{\lambda>0}\lambda B$ with the norm defined by B is a closed subspace of an α-WCG Banach space. Let q be the quotient map of E onto $E/\bigcap_{\lambda>0}\lambda B$. The graph of q is closed in $E \times F$ (see e.g. [37], Ch. VI, Proof of Proposition 11), so q is continuous by hypothesis. The closed unit ball of F is $cl(q(B))$, and the closed unit ball of $E/\bigcap_{\lambda>0}\lambda B$ is $q(B)$. Hence $q^{-1}(cl(q(B))) = q^{-1}(q(B)) = B$ is a neighbourhood of 0 in E, thus E is $M(\alpha)$-barrelled.

Necessity: Suppose E is $M(\alpha)$-barrelled and let t be a linear mapping with closed graph of E into a closed subspace F of an α-WCG Banach space. E is $G(\alpha)$-barrelled and $F \in \mathcal{R}(\alpha)$. Hence by Theorem 2.1, t is weakly continuous. Thus if A is the closed unit ball of F, $B = t^{-1}(A)$ is a barrel in E. Now the kernel of t is $\bigcap_{\lambda>0}\lambda B$ and so we can define a one-to-one linear mapping s of $E/\bigcap_{\lambda>0}\lambda B$ into F by putting $s(x + \bigcap_{\lambda>0}\lambda B) = t(x)$ $(x \in E)$. If $E/\bigcap_{\lambda>0}\lambda B$ has the norm topology defined by B, then s is an isometry of $E/\bigcap_{\lambda>0}\lambda B$ onto a subspace of F. It follows that B is an $M(\alpha)$-barrel in E and therefore an ξ-neighbourhood of 0. Consequently t is continuous. Δ

3.5 Corollary

Let $E[\xi]$ be a $G(\alpha)$-barrelled space with dual E'. The set of $M(\alpha)$-barrels in $E[\xi]$ forms a base of neighbourhoods of 0 for a topology η of the dual pair (E, E'). A topology of the dual pair (E, E') is $M(\alpha)$-barrelled if and only if it is finer than η.

Proof

We have already noted that the set of $M(\alpha)$-barrels in $E[\xi]$ forms a base of neighbourhoods of 0 for a locally convex topology on E which is finer than $\sigma(E, E')$. Theorems 2.1 and 3.4 show that $E[\tau(E, E')]$ is $M(\alpha)$-barrelled. Thus we also have that η is coarser than $\tau(E, E')$ and therefore η is a topology of the dual pair (E, E'). The last part is immediate. Δ

Note that η may be strictly finer than ξ (see comment at the end of Example 4.3).

3.6 Corollary

Let $t : E \to F$ be a linear mapping with closed graph of an $M(\alpha)$-barrelled space E into an α-WCG B_r-complete locally convex space. Then t is continuous.

Proof

For each neighbourhood U of 0 in F, $U°$ is $\sigma(F', F)$-compact; hence as in the proof of Theorem 2.1 we can show that $X = U° \cap t'^{-1}(E')$ is $\sigma(F', F)$-closed. It follows that $t'^{-1}(E')$ is a nearly closed dense subspace of F'; so it is $\sigma(F', F)$-closed since F is B_r-complete. Thus $t'(F') \subseteq E'$ and t is weakly continuous. Now let U be a closed absolutely convex neighbourhood of 0 in F and let q be the quotient map of F onto $F/\bigcap_{\lambda>0}\lambda U$. Let G be the completion of $F/\bigcap_{\lambda>0}\lambda U$ under the norm defined by U. Regarding q as a mapping into G we have that $q \circ t$ is weakly continuous and therefore its graph is closed. But G is an α-WCG Banach space and so by Theorem 3.4, $q \circ t$ is continuous. Since F is the projective limit of the spaces G by the mappings q when U runs through a base of closed absolutely convex neighbourhoods of 0 in F , we deduce that t is continuous. △

As a consequence of Theorem 3.4 the following permanence properties of M(α)-barrelled spaces follow immediately on applying Theorems 2.1 and 2.2 of [18] and of [33], Theorem 3, Corollary 1.

3.7 Theorem

(a) An inductive limit of M(α)-barrelled spaces is M(α)-barrelled.

(b) Any product of M(α)-barrelled spaces is M(α)-barrelled.

(c) If E is an M(α)-barrelled space and G is any subspace of the completion of E which contains E , then G is also M(α)-barrelled.

From Theorem 3.4 and Corollary 3.6 we see that the M(α)-barrelled spaces may be characterised as those locally convex spaces which can serve as domain spaces in a closed graph theorem where the range space is an arbitrary element of the class $\mathcal{R}'(\alpha)$ of α-WCG B_r-complete (or B-complete) locally convex spaces. Now if $F \in \mathcal{R}'(\alpha)$, so also does $F \times \phi$. For if $D_i (i \in I)$ is a family of $\sigma(F, F')$-compact absolutely convex subsets of F whose union is total in F , where $card(I) \le \alpha$, and if $\{d_n : n \in \mathbb{N}\}$ is a dense subset of ϕ , then $D = \{D_i \times \{\lambda d_n : |\lambda| \le 1\} : i \in I, n \in \mathbb{N}\}$ is a family of weakly compact absolutely convex subsets of $F \times \phi$ such that $\bigcup\{D_i \times \{\lambda d_n : |\lambda| \le 1\} : i \in I, n \in \mathbb{N}\}$ is total in $F \times \phi$ and $card(D) \le \alpha \aleph_0 = \alpha$. Thus by Theorem 2.2 of Chapter III we have:

3.8 Theorem

A countable codimensional subspace of an M(α)-barrelled space is again M(α)-barrelled.

§4. Examples

Our first example shows that an $M(\alpha)$-barrelled space need not be a Mackey space or an α-barrelled space. We make use of the following extension of a result mentioned in the last paragraph of [29].

4.1 Lemma

If I is an index set with cardinality greater than α, then $\ell_1(I)$ cannot be embedded in an α-WCG Banach space.

Proof

Suppose there is an α-WCG Banach space E which contains $\ell_1(I)$. Let q be the quotient map of E' onto $E'/(\ell_1(I))^\circ = \ell_\infty(I)$. Let S be the closed unit ball of $\ell_\infty(I)$ and put $S_\alpha = \{(\xi_i) \in S : |\{i : \xi_i \neq 0\}| \leq \alpha\}$. Then S_α is $\sigma(\ell_\infty(I), \ell_1(I))$-dense in S. By the Hahn-Banach theorem we can find an absolutely convex equicontinuous subset T_α of E', namely the intersection of $q^{-1}(S_\alpha)$ with the closed unit ball of E', such that $q(T_\alpha) = S_\alpha$. Then if T is the $\sigma(E', E)$-closure of T_α we have that T is $\sigma(E', E)$-compact and hence $q(T)$ is $\sigma(\ell_\infty(I), \ell_1(I))$-compact. Therefore $q(T)$ is closed and $S_\alpha \subseteq q(T)$ which implies that $S = \text{cl}\, S_\alpha \subseteq q(T)$. On the other hand

$$q(T) = q(\text{cl}(T_\alpha)) \subseteq \text{cl}(q(T_\alpha)) = \text{cl}\, S_\alpha = S .$$

Thus $q(T) = S$.

For any subset C of T_α with cardinality at most α, we can find a subset I' of I with cardinality at most α such that $q(C) \subseteq \{(\xi_i) \in S : \xi_i = 0 \text{ if } i \notin I'\}$. But this last set is $\sigma(\ell_\infty(I), \ell_1(I))$-closed and so it follows that the $\sigma(E', E)$-closure of C is contained in T_α. We now deduce from [48], Theorem 1 that T_α is $\sigma(E', E)$-compact. This implies that $S_\alpha = q(T_\alpha)$ is $\sigma(\ell_\infty(I), \ell_1(I))$-compact and therefore $S_\alpha = S$, which is false if the cardinality of I is greater than α.

4.2 Example

Let $E = \mathbb{R}^{(I)}$ (the topological direct sum of I - copies of the scalar field $\mathbb{R}$) and $E' = \mathbb{R}^I$ where the cardinality of I is 2^α. Since $E[\tau(E, E')]$ is barrelled we have that $E[\eta]$ is $M(\alpha)$-barrelled where η is the topology of Corollary 3.5. Let $B = ([-1, 1]^I)^\circ$. The Banach space obtained by completing $E/\bigcap_{\lambda>0}\lambda B$ ($\simeq E$) is $\ell_1(I)$. It follows then from Lemma 4.1 that B is not an $M(\alpha)$-barrel and so by (iii) of 3.2, B is not an η-neighbourhood of 0. Since B is a $\tau(E, E')$-neighbourhood of 0 we therefore have that η is strictly coarser than $\tau(E, E')$.

By [14], Theorem, $[-1, 1]^I$ has a $\sigma(E', E)$-dense subset D of cardinality α. Consequently $E[\eta]$ cannot be α-barrelled since otherwise $D^\circ = ([-1, 1]^I)^\circ = B$ would be an η-neighbourhood of 0.

Finally, we give an example of a space which is α-barrelled and therefore also $G(\alpha)$-barrelled ([32], p.251, Remarks), but which is not $M(\alpha)$-barrelled.

4.3 Example

Consider the Hilbert space $\ell_2(I)$ where $\text{card}(I) > \alpha$. The set $\mathcal{U}$ of polars of the $\sigma(\ell_2(I), \ell_2(I))$-bounded sets with cardinality at most α forms a base of neighbourhoods of 0 for some topology λ; for the elements of $\mathcal{U}$ satisfy the conditions of Theorem 3.1 of Ch. I. To see this, let $\mathcal{A}$ be the set of all $\sigma(\ell_2(I), \ell_2(I))$-

bounded subsets of $\ell_2(I)$ with cardinality at most α ; then

(i) let $A°, B° \in \mathcal{U}$ with $A, B \in \mathcal{A}$; since

$$\text{card}(A \cup B) = \text{card}(A) + \text{card}(B) \le \alpha + \alpha = \alpha ,$$

it follows that $A \cup B \in \mathcal{A}$ and hence $A° \cap B° = (A \cup B)° \in \mathcal{U}$;

(ii) let $A° \in \mathcal{U}$ with $A \in \mathcal{A}$ and $\lambda \neq 0$; then

$$\frac{1}{|\lambda|} A \in \mathcal{A}$$

which implies that

$$(\frac{1}{|\lambda|} A)° = |\lambda| \, A° \in \mathcal{U} ;$$

since $A°$ is balanced we have $\lambda A° = |\lambda| A° \in \mathcal{U}$;

(iii) for each $A \in \mathcal{A}$, $A°$ is clearly absolutely convex and absorbent.

$\mathcal{U}$ defines a topology ξ of the dual pair $(\ell_2(I), \ell_2(I))$ under which $\ell_2(I)$ is α-barrelled.

Now since $\text{card}(I) > \alpha$, ξ is strictly coarser than the norm topology on $\ell_2(I)$. Suppose ξ coincides with the norm topology of $\ell_2(I)$. Then in the dual space, the closed unit ball of $\ell_2(I)$ must be contained in the $\sigma(\ell_2(I), \ell_2(I))$-closed absolutely convex envelope of a set with cardinality at most α . This leads to a contradiction as in Lemma 4.1. It follows that the closed unit ball of $\ell_2(I)$ is not a ξ-neighbourhood of 0 , but it is an $M(\alpha)$-barrel in $\ell_2(I)[\xi]$.

Note that the identity map of $\ell_2(I)[\xi]$ onto $\ell_2(I)$ with norm topology is weakly continuous but not continuous. This shows that weak continuity cannot be replaced by continuity in Theorem 2.1.

CHAPTER V

ON TOPOLOGICAL ALGEBRAS

§1. Introduction

Baker [3] and Rosa [38] used the concept of B-completeness (and B_r-completeness) to give closed graph theorems for topological groups and topological algebras. Sulley in [43] studies more about B- and B_r-complete topological abelian groups.

In this chapter our objects of study are topological algebras. The algebras we are talking about are not necessarily commutative; but by an ideal of an algebra we shall always mean a 2-sided ideal.

1.1 Definitions

A locally convex algebra E is an algebra over $\mathbb{K}$ together with a Hausdorff locally convex topology which makes multiplication jointly continuous.

A locally convex algebra E is called B-complete (B_r-complete) if every continuous (continuous, one-to-one) and nearly open algebra homomorphism of E onto a locally convex algebra F is open.

We have to point out that many authors just require separate continuity in the definition of a locally convex algebra; but since we need to make use of completion this is insufficient for our purposes. So following Rosa [38] we suppose the multiplications of our algebras to be jointly continuous.

It is easily seen from the definitions that every B-complete algebra is B_r-complete and that the quotient of a B-complete algebra modulo a closed ideal is B-complete. A locally convex algebra which is a B-complete space is a B-complete algebra, in particular, Banach algebras and complete metrizable locally convex algebras are B-complete. A B_r-complete (even a complete B_r-complete) algebra need not be B-complete ([38], Example 4.3). Unlike the topological vector space case, a B-complete (B_r-complete) algebra need not be complete ([38], p.202); but as the following Theorem shows, the completions of B- and B_r-complete algebras are of the same type. Rosa ([38], Theorem 3.2) showed that $C(X)$ is B-complete if and only if X is a k-space. A closed subalgebra of a B-complete algebra, also the quotient of a B_r-complete algebra need not be B_r-complete; B-complete (B_r-complete) algebras are not productive ([38], p.206). Every B-complete space is a B-complete algebra if we take multiplication to be zero ($xy = 0$ for all x, y). But a B-complete algebra need not be a B-complete space, for example if X is a k-space which is not normal, then $C(X)$ is a B-complete algebra but not a B-complete space by Corollary 3.3 of [38].

The following theorem was originally proved by Sulley [43] for topological groups. Rosa [38] pointed out that the same criterion can be adapted for topological algebras. We shall use it as a key theorem for most of our results in this chapter.

1.2 Theorem ([38], Theorem 2.4)

Let G be a dense subalgebra of a locally convex algebra E.

(a) G is a B_r-complete algebra if and only if E is B_r-complete and G has non-zero intersection with every non-zero closed ideal of E.

(b) G is a B-complete algebra if and only if E is B-complete and $G \cap I$ is dense in I for every closed ideal I of E.

In ϕ it is possible to define various multiplications such as coordinatewise, convolution, Dirichlet and zero multiplications and many others (see the Table of multiplications on ϕ in Section 5). All multiplications on ϕ are jointly continuous making ϕ into a locally convex algebra. For, since ϕ has its finest locally convex topology, each multiplication on ϕ being bilinear is separately continuous. We know that ϕ is a barrelled space (Ch. I, Section 4, c)). Also ϕ , being the strong dual of the metrizable locally convex space ω is a DF-space ([13], Ch.4, Part 3, Theorem 1). Hence by [13], Ch.4, Part 3, Theorem 2, Corollary 1, the multiplication is jointly continuous (or see [26], §40, 5(3)).

For a locally convex algebra E and a suitable multiplication on ϕ it is possible to define various multiplications on $E \times \phi$ to make it into a locally convex algebra. In Sections 2 and 4 we shall define two such multiplications on $E \times \phi$, one the natural pointwise multiplication, the other an extension of the unitization process. We discuss B- and B_r-completeness of $E \times \phi$ when E is so in each case. In 5.3 we shall give an example of a multiplication on $E \times \phi$ which extends the multiplication of E but which is not even separately continuous.

In this chapter we shall use different forms of Lemmas 2.1 and 2.2 of Chapter II. So in order to cover all the cases needed, we restate them in the following way.

Let E be a locally convex algebra and let $\Phi = \phi$ or $\mathbb{K}^n$ with its finest locally convex topology. Suppose $E \times \Phi$ has a multiplication which induces the given multiplication on E . Let t be a continuous nearly open algebra homomorphism of $E \times \Phi$ onto a locally convex algebra F .

1.3 Lemma

(a) The restriction of t to E is a continuous nearly open algebra homomorphism of E onto $t(E)$.

(b) Suppose further that $t(E)$ is closed in F . If either E is a B-complete algebra, or E is a B_r-complete algebra and $t|_E$ is one-to-one, then t is open.

Proof

(a) Since E is a subalgebra of $E \times \phi$ the restriction of t on E is an algebra homomorphism. Regarding t as a linear mapping of the locally convex space $E \times \phi$ onto the locally convex space F , the result follows from Lemma 2.1 of Chapter II. In the case $E \times \mathbb{K}^n$, the codimension of $t(E)$ in F would be finite and the proof is almost the same as for the case $E \times \phi$.

(b) If E is B-complete or, E is B_r-complete and $t|_E$ is one-to-one, it follows that $t|_E$ is open as a mapping onto $t(E)$. If $t(E)$ is closed in F we deduce as in Lemma 2.2 of Chapter II that F is the locally convex direct sum of $t(E)$ and any supplement H . It now follows as in the proof of Theorem 2.4 of Chapter II that t is open. Δ

§2. B- and B_r-completeness of $E \times \phi$

Let E be a locally convex algebra and let Φ have any multiplication. Then the pointwise multiplication on $E \times \Phi$ defined by

$$(x, (\lambda_n))(y, (\mu_n)) = (xy, (\lambda_n)(\mu_n))$$

for all $(x, (\lambda_n)), (y, (\mu_n)) \in E \times \Phi$, is obviously a jointly continuous multiplication on $E \times \Phi$ and $E \times \Phi$ is a locally convex algebra under this multiplication containing E as a subalgebra (in fact E is an ideal).

Throughout this section multiplication on $E \times \Phi$ will be pointwise.

<u>2.1 Lemma</u>

Let E be a B_r-complete algebra. If $x \in \hat{E}$ and $xy = yx = 0$ for all $y \in E$, we must have $x \in E$.

<u>Proof</u>

The Lemma is trivial if E is complete. So let E be incomplete and suppose there exists an $x_0 \in \hat{E} \setminus E$ such that $x_0 y = y x_0 = 0$ for all $y \in E$. Then $I = \{\lambda x_0 : \lambda \in \mathbb{K}\}$ is a closed ideal in $\hat{E}$; I being a one-dimensional subspace of $\hat{E}$ is obviously closed and for each $y \in \hat{E}$, if (y_α) is a net in E which converges to y, then we have $y(\lambda x_0) = \lim y_\alpha(\lambda x_0) = 0 \in I$ and $(\lambda x_0)y = \lim(\lambda x_0)y_\alpha = 0 \in I$. But $I \cap E = \{0\}$ which is impossible by Theorem 1.2. Δ

<u>2.2 Lemma</u>

Let E be a B_r-complete algebra. Let t be a continuous nearly open algebra homomorphism of $E \times \Phi$ onto a locally convex algebra F such that $t|_E$ is one-to-one. Then $t(E)$ is closed in F.

<u>Proof</u>

By Lemma 1.3 (a), $t|_E$ is nearly open, hence open by hypothesis. Therefore $t|_E$ is a topological isomorphism of E onto $t(E)$. If E is complete, then $t(E)$ would also be complete, hence closed in F. Suppose E is not complete.

Let $\hat{t}$ be the extension of t by continuity to a homomorphism of $\hat{E} \times \Phi$ into $\hat{F}$. We show that $\ker t = \ker \hat{t}$. Let $(x, (\lambda_n)) \in \ker \hat{t}$. For all $y \in \hat{E}$ we have

$$\hat{t}(xy, 0) = \hat{t}((x, (\lambda_n))(y, 0)) = \hat{t}(x, (\lambda_n))\hat{t}(y, 0) = 0 .$$

Let $\widehat{t|_E}$ be the extension of $t|_E$ by continuity to a homomorphism of $\hat{E}$ into $\widehat{t(E)}$. Then $\widehat{t|_E} = \hat{t}|_{\hat{E}}$ is an isomorphism of $\hat{E}$ onto $\hat{t}(\hat{E})$, ([37], Ch. VI, Proposition 6, Corollary 1), and it follows that $xy = 0$ for all $y \in \hat{E}$. Similarly $yx = 0 \ \forall y \in \hat{E}$. Then by Lemma 2.1 we have $x \in E$, that is, $(x, (\lambda_n)) \in \ker t$.

Let $y \in \hat{t}(\hat{E}) \cap F$. Since t is onto, there are $\hat{x} \in \hat{E}$, $(x, (\lambda_n)) \in E \times \Phi$ such that $t(x, (\lambda_n)) = y = \hat{t}(\hat{x}, 0)$. Hence $\hat{t}(x - \hat{x}, (\lambda_n)) = 0$, that is, $(x - \hat{x}, (\lambda_n)) \in \ker \hat{t} = \ker t$ which implies that $(x - \hat{x}, (\lambda_n)) \in E \times \Phi$. So $x - \hat{x}$ and hence $\hat{x}$ belong to E, from which we get $y = \hat{t}(\hat{x}, 0) = t(\hat{x}, 0) \in t(E)$. Since also $t(E) \subseteq \hat{t}(\hat{E}) \cap F$ we have $t(E) = \hat{t}(\hat{E}) \cap F$. But $\hat{t}(\hat{E})$ is closed in $\hat{F}$ by the first part of the proof, therefore $\hat{t}(\hat{E}) \cap F = t(E)$ is closed in F. Δ

<u>2.3 Corollary</u>

Let E be a B-complete algebra. Let t be a continuous nearly open algebra homomorphism of $E \times \Phi$ onto a locally convex algebra F. Then $t(E)$ is closed in F.

<u>Proof</u>

Put $J = \{(x, 0) : t(x, 0) = 0\}$ and $I = \{x : (x, 0) \in J\}$. Clearly J, I are closed ideals in $E \times \Phi$ and E respectively. Also it is easily seen that

$h : (E \times \Phi)/J \to (E/I) \times \Phi$ defined by $h((x, (\lambda_n)) + J) = (x + I, (\lambda_n))$

is a topological isomorphism (for the proof and other variants of this see Lemmas 3.5 and 4.5). Now define $s : (E/I) \times \Phi \to F$ by

$$s(x + I, (\lambda_n)) = t(x, (\lambda_n)) \qquad (x \in E, (\lambda_n) \in \Phi) .$$

In fact, if $q : E \times \Phi \to (E \times \Phi)/J$ is the quotient map, we have $t = s \circ h \circ q$. s is clearly a well-defined algebra homomorphism. Continuity of s follows from the continuity of t and openness of h and q . s is also nearly open. For a neighbourhood W of 0 in $(E/I) \times \Phi$ there are neighbourhoods of 0 in E and Φ say U and V respectively such that $(U + I) \times V \subseteq W$. Since

$$\begin{aligned} \mathrm{cls}((U + I) \times V) &= \mathrm{cls}(h(U \times V + J)) \\ &= \mathrm{cls}(h(q(U \times V))) = \mathrm{clt}(U \times V) , \end{aligned}$$

and since t is nearly open, then $\mathrm{cls}(W)$ which contains $\mathrm{clt}(U \times V)$ is a neighbourhood of 0 in F . Finally s is one-to-one on E/I ; let $(x + I, 0) \neq 0$, then $x \notin I$. Therefore $t(x, 0) \neq 0$ and hence $s(x + I, 0) \neq 0$.

Now E/I is a B_r-complete algebra and s is a continuous nearly open algebra homomorphism of $(E/I) \times \Phi$ onto F which is one-to-one on E/I . So $s(E/I) = t(E)$ is closed in F by Lemma 2.2. Δ

2.4 Theorem

If E is a B-complete (B_r-complete) algebra, then so is $E \times \Phi$.

Proof

Let t be a continuous (continuous, one-to-one), nearly open algebra homomorphism of $E \times \Phi$ onto an arbitrary locally convex algebra F . By Corollary 2.3 (Lemma 2.2) $t(E)$ is closed in F and hence by Lemma 1.3(b), t is open. Δ

§3. B- and B_r-completeness of the unitization of an algebra

Recall that the unitization of an algebra E is the algebra $E_1 = \{(x, \lambda) : x \in E, \lambda \in \mathbb{K}\}$ with coordinatewise addition and scalar multiplication and multiplication defined by

$$(x, \lambda)(y, \mu) = (xy + \lambda y + \mu x, \lambda\mu), \quad (x, y \in E, \lambda, \mu \in \mathbb{K}) .$$

As a vector space E_1 is just $E \times \mathbb{K}$. When E is a locally convex algebra we therefore give E_1 the product topology and it is easily seen that E_1 is then a locally convex algebra.

Although the purpose of unitization is to adjoin a unit to an algebra without a unit, it is still meaningful in the present context to apply it to an algebra with a unit. The original unit is then no longer a unit.

For an algebra E, by E_1 we shall always mean the unitization of E .

From the following easy Proposition we see that if the unitization of a locally convex algebra is B-complete (B_r-complete), so also is the algebra itself. But in general, B-completeness (B_r-completeness) of a locally convex algebra E does not imply B-completeness (B_r-completeness) of its unitization (Example 3.2). In this section we shall give necessary and sufficient conditions on B- and B_r-complete algebras E to make E_1 of the same kind.

3.1 Proposition

Let E be a locally convex algebra such that E_1 is B-complete (B_r-complete). Then E is B-complete (B_r-complete).

Proof

Let t be a continuous (continuous, one-to-one) nearly open algebra homomorphism of E onto a locally convex algebra F. Define the mapping t_1 of E_1 onto F_1 by

$$t_1(x, \lambda) = (t(x), \lambda) \qquad (x \in E, \lambda \in \mathbb{K}) .$$

t_1 is clearly well-defined and linear. For $x, y \in E$, $\lambda, \mu \in \mathbb{K}$

$$\begin{aligned} t_1((x, \lambda)(y, \mu)) &= t_1(xy + \lambda y + \mu x, \lambda\mu) \\ &= (t(xy + \lambda y + \mu x), \lambda\mu) \\ &= (t(x), \lambda)(t(y), \mu) \\ &= t_1(x, \lambda)\, t_1(y, \mu) . \end{aligned}$$

Hence t_1 is an algebra homomorphism. If t is one-to-one, so also is t_1; for, if $t_1(x, \lambda) = 0$, then $(t(x), \lambda) = 0$ which implies that $t(x) = 0$ and $\lambda = 0$; thus $x = 0$ and $\lambda = 0$. Therefore $(x, \lambda) = 0$. t_1 is continuous; for, if W is a neighbourhood of 0 in F_1, there are neighbourhoods U, V of 0 in F and $\mathbb{K}$ respectively such that $U \times V \subseteq W$. Since t is continuous, $t_1^{-1}(U \times V) = t^{-1}(U) \times V$ is a neighbourhood of 0 in E_1 and we have $t_1^{-1}(U \times V) \subseteq t_1^{-1}(W)$. Also if W is a neighbourhood of 0 in E_1 there are neighbourhoods U, V of 0 in E and $\mathbb{K}$ respectively such that $U \times V \subseteq W$. Now, $(\mathrm{cl}\,t(U)) \times V$ is a neighbourhood of 0 in F_1 and we have

$$(\mathrm{cl}\,t(U)) \times V \subseteq \mathrm{cl}(t(U) \times V) = \mathrm{cl}(t_1(U \times V)) \subseteq \mathrm{cl}\,t_1(W) .$$

Hence t_1 is nearly open. It follows from the hypothesis that t_1

and hence t are open. Δ

The following example shows that the unitization of a B-complete algebra need not be even B_r-complete.

<u>3.2 Example</u>

Let $\mathcal{K}$ be the algebra of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ with compact support. We know that $\mathcal{K}$ is a dense subalgebra of $C(\mathbb{R})$ equipped with the compact-open topology. $C(\mathbb{R})$ being a Fréchet space is clearly a B-complete locally convex algebra. We show that $\mathcal{K}$ is also B-complete.

Let I be a non-zero closed ideal in $C(\mathbb{R})$. (There exists a closed subset A of $\mathbb{R}$ such that $I = \{f \in C(\mathbb{R}) : f(A) = \{0\}\}$. ([30], Theorem 2.1)). Let $f \in I$ and let

$$N(K, f, \varepsilon) = \{g \in C(\mathbb{R}) : |f(x) - g(x)| \le \varepsilon \, , \; \forall \; x \in K\}$$

be a neighbourhood of f, where K is a compact subset of $\mathbb{R}$ and $\varepsilon > 0$. Let B be a compact subset of $\mathbb{R}$ such that K is contained in the interior of B. Then by Urysohn's Lemma (or by an elementary way) we can find a $g \in \mathcal{K}$ such that $g(K) = \{1\}$ and $g(x) = 0$ for all $x \in \mathbb{R} \backslash \mathrm{Int}\, B$. Now fg which belongs to $I \cap \mathcal{K}$ is also in $N(K, f, \varepsilon)$. For, $\forall \; x \in K$ we have $|f(x) - (fg)(x)| = |f(x)|\,|1 - g(x)| = 0 < \varepsilon$. This shows that $I \cap \mathcal{K}$ is dense in I. For the case $I = \{0\}$, obviously $I \cap \mathcal{K} = I$. Hence the B-completeness of $\mathcal{K}$ follows from Theorem 1.2(b).

Now we have $\hat{\mathcal{K}} = C(\mathbb{R})$ and $\hat{\mathcal{K}}_1 = \hat{\mathcal{K}} \times \mathbb{K}$, ($\hat{\mathcal{K}}_1$ means the completion of $\mathcal{K}_1$). $\hat{\mathcal{K}}_1$ being a complete metrizable locally convex algebra is B-complete, hence B_r-complete. But $\mathcal{K}_1$ is not

B_r-complete. For, if e is the identity element of $\hat{\mathcal{K}} = C(\mathbb{R})$, then $I = \{\lambda(e, -1) : \lambda \in \mathbb{K}\}$ is a closed ideal in $\hat{\mathcal{K}}_1 = \hat{\mathcal{K}} \times \mathbb{K}$. Clearly I is a closed subspace of $\hat{\mathcal{K}}_1$ and for $(f, \mu) \in \hat{\mathcal{K}}_1$, $\lambda(e, -1) \in I$ we have

$$(f, \mu)(\lambda e, -\lambda) = (\lambda e, -\lambda)(f, \mu)$$
$$= (\lambda ef - \lambda f + \mu\lambda e, -\mu\lambda) = \mu\lambda(e, -1) \in I .$$

But $I \cap \mathcal{K}_1 = (0, 0)$, for, if $(\lambda_0 e, -\lambda_0) \in I \cap \mathcal{K}_1$ for some $\lambda_0 \in \mathbb{K}$, we must have $\lambda_0 e \in \mathcal{K}$ which is true only if $\lambda_0 = 0$. Hence by theorem 1.2(a), $\mathcal{K}_1$ can not be B_r-complete.

Now we give a necessary and sufficient condition for a B_r-complete algebra which guarantees B_r-completeness of its unitization.

<u>3.3 Theorem</u>

Let E be a B_r-complete algebra. Then E_1 is B_r-complete if and only if:

either $\hat{E}$ has no identity,

or E has an identity.

<u>Proof</u>

If $\hat{E}$ has an identity e which is not in E, then as in the last paragraph of Example 3.2, $\{\lambda(e, -1) : \lambda \in \mathbb{K}\}$ is a non-zero ideal of $\hat{E}_1$ whose intersection with E_1 consists only of the zero element. Thus if E_1 is B_r-complete, then $\hat{E}$ has no identity unless it is already in E.

For the converse let t be a one-to-one, continuous nearly open algebra homomorphism of E_1 onto an arbitrary locally convex algebra F. $t|_E$ is a one-to-one, continuous algebra homomorphism of E onto the subalgebra $G = t(E)$ of F. $t|_E$ is also nearly open (Lemma 1.3(a)), hence open by hypothesis. Therefore $t|_E$ is a topological isomorphism of E onto G. If we show that G is closed in F, it will follow by Lemma 1.3(b) that t is open. Let $s : \hat{E} \to \hat{G}$ be the extension of $t|_E$ by continuity. s is a topological isomorphism of $\hat{E}$ onto $s(\hat{E})$ by [37], Ch. VI, Proposition 6, Corollary 1. But, since $s(\hat{E})$ is a complete locally convex algebra with $G \subseteq s(\hat{E}) \subseteq \hat{G}$, by uniqueness of the completion we have $s(\hat{E}) = \hat{G}$, i.e. s is an isomorphism onto $\hat{G}$.

If G is not closed in F, since $\hat{G}$ is closed in $\hat{F}$ and hence $\hat{G} \cap F$ is closed in F, G should be strictly contained in $\hat{G} \cap F$. But, as G is one codimensional in F, we must then have $\hat{G} \cap F = F$. Hence $F \subseteq \hat{G}$. Now if e denotes the identity element of E_1, then $s^{-1}(t(e))$ would be an identity in $\hat{E}$. We see this as follows. Let $x \in \hat{E}$. There exists a unique $y \in \hat{G}$ such that $x = s^{-1}(y)$. Now if (x_α) is a net in E which converges to x, we have

$$\begin{aligned} s^{-1}(t(e))x &= s^{-1}(t(e))s^{-1}(y) \\ &= s^{-1}(t(e)y) \\ &= s^{-1}(t(e)s(x)) \\ &= \lim s^{-1}(t(e)t(x_\alpha)) \\ &= \lim s^{-1}(t(x_\alpha)) \\ &= \lim s^{-1}s(x_\alpha) \\ &= \lim x_\alpha = x . \end{aligned}$$

Similarly $xs^{-1}(t(e)) = x$.

If $\hat{E}$ has no identity then we clearly have a contradiction. Suppose E has identity e_1. Since e_1 must be the identity element of $\hat{E}$ we therefore have $e_1 = s^{-1}t(e)$, i.e. $s(e_1) = t(e)$ or equivalently $t(e_1) = t(e)$ since $e_1 \in E$. This contradicts the fact that t is one-to-one since $e_1 \neq e$. It follows that G is closed in both cases. Δ

3.4 Corollary

If E is a complete B_r-complete algebra, then E_1 is B_r-complete.

Proof

Since $E = \hat{E}$, either E has an identity or $\hat{E}$ (= E) has no identity. Δ

Remark

If E has no identity but $\hat{E}$ has an identity e, then the natural way of adjoining an identity to E would be to form the subalgebra of $\hat{E}$ generated by E and e. From Theorem 1.2(a), this will be B_r-complete if E is B_r-complete. If $\hat{E}$ has no identity, then we have to use unitization and by Theorem 3.3, E_1 is B_r-complete if E is B_r-complete.

Our next theorem establishes a necessary and sufficient condition on a B-complete algebra which ensures B-completess of its unitization. First a lemma.

3.5 Lemma

Let E be a locally convex algebra. Let I be a non-empty subset of E and let $J = \{(x, 0) : x \in I\} \subseteq E_1$. Then I is a closed ideal of E if and only if J is a closed ideal of E_1.

In this case E_1/J is topologically isomorphic with $(E/I)_1$.

A more general case of this Lemma and its proof have been given later in Lemma 4.5. Therefore we omit the proof of this Lemma here.

3.6 Theorem

Let E be a B-complete algebra. Then E_1 is B-complete if and only if for every closed ideal I in E

either $\widehat{E/I}$ has no identity,

or E/I has an identity.

Proof

Let E_1 be B-complete. Suppose there exists a closed ideal I in E such that $\widehat{E/I}$ has an identity while E/I has no identity. Since E/I is a B_r-complete algebra, it follows from Theorem 3.3 that $(E/I)_1$ can not be B_r-complete. Now $J = I \times \{0\}$ is a closed ideal in E_1 and hence by Lemma 3.5, E_1/J is not B_r-complete which contradicts the fact that E_1 is B-complete.

For the converse, let t be a continuous nearly open algebra homomorphism of E_1 onto a locally convex algebra F . Suppose there exists an element (x_0, λ_0) in $J = \ker t$ with $\lambda_0 \neq 0$. Since $t(x_0, \lambda_0) = t(x_0, 0) + \lambda_0 t(0, 1) = 0$ we have $t(0, 1) = -\frac{1}{\lambda_0} t(x_0, 0) \in t(E)$. Therefore for all $x \in E, \lambda \in \mathbb{K}$,

$$t(x, \lambda) = t(x, 0) + \lambda t(0, 1) \in t(E) ,$$

which implies that $t(E) = F$. Now $t|_E$ is a continuous nearly open (Lemma 1.3(a)) algebra homomorphism of E onto F . By hypothesis $t|_E$ is open, hence t is open. On the other hand if $\lambda = 0$ for all $(x, \lambda) \in J = \ker t$, $I = \{x : (x, 0) \in J\}$ is a closed ideal in E by Lemma 3.5 and we have $E_1/J \simeq (E/I)_1$. E/I is a B_r-complete algebra and if, either $\widehat{E/I}$ has no identity or E/I has an identity, it follows from Theorem 3.3 that $(E/I)_1$ i.e. E_1/J is B_r-complete. Let $t_1 : E_1/J \to F$ be the continuous one-to-one nearly open algebra homomorphism defined by t . Since E_1/J is B_r-complete, then t_1 and hence t are open. This completes the proof. Δ

3.7 Corollary

Let E be a B-complete algebra with an identity. Then E_1 is B-complete.

The following is an example of a B-complete algebra E with closed ideals I_1, I_2 such that $\widehat{E/I_1}$ has no identity and E/I_2 has an identity.

3.8 Example

Suppose $C(X)$ is a B-complete algebra with $\text{card}(X) \geq 2$. By [38], Theorem 3.2, X is a k-space. Let $x_0 \in X$ and let $A = \{f \in C(X) : f(x_0) = 0\}$. A is clearly a locally convex algebra. Further, we show that the unitization A_1 of A is $C(X)$.

Define $h : A_1 \to C(X)$ by $h(f, \lambda) = f + \lambda e$ for all $(f, \lambda) \in A_1$, where e is the identity element of $C(X)$. h is obviously an algebraic isomorphism. To show that h is continuous, let $U = \{f \in C(X) : |f(x)| \leq \varepsilon \ \forall\ x \in K\}$ be a neighbourhood of 0 in $C(X)$, where K is a compact subset of X and $\varepsilon > 0$. Then $\{f \in A : |f(x)| \leq \frac{\varepsilon}{2} \ \forall\ x \in K\}$ is a neighbourhood of 0 in A and we have

$$h^{-1}(U) = \{(f, \lambda) \in A_1 : f + \lambda e \in U\}$$
$$= \{(f, \lambda) \in E_1 : |(f + \lambda e)(x)| \leq \varepsilon \quad \forall\, x \in K\}$$
$$\supseteq \{f \in A : |f(x)| \leq \tfrac{\varepsilon}{2} \quad \forall\, x \in K\} \times \{\lambda \in \mathbb{K} : |\lambda| \leq \tfrac{\varepsilon}{2}\} \quad .$$

A is a closed maximal ideal in $C(X)$ corresponding to the closed subset $\{x_0\}$ of X, ([30], Corollary 2.2). Therefore $C(X)$ is the topological direct sum of A and the linear subspace of $C(X)$ generated by e ([37], Ch. V, Proposition 29, Corollary). Now if W is a neighbourhood of 0 in A_1, there are neighbourhoods U, V of 0 in A and $\mathbb{K}$, respectively, such that $U \times V \subseteq W$ and $h(W) \supseteq h(U \times V) = U + Ve$. Since $U + Ve$ is a neighbourhood of 0 in $C(X)$, it follows that h is open.

We conclude that A_1 and hence by Proposition 3.1, A are B-complete.

Let I be a proper closed ideal in A. I is also a closed ideal in $C(X)$. For, if $g \in I$, $f \in C(X)$, then $fg = (f - f(x_0)e)g + f(x_0)g \in AI + I \subseteq I$. Hence by [30], Theorem 2.1, there exists a closed subset Y of X such that

$$I = I_Y = \{f \in C(X) : f(Y) = \{0\}\} .$$

As on page 204 of [38] we have

$$C(X)/I = \{f|_Y : f \in C(X)\} \quad .$$

Similarly

$$A/I = \{f|_Y : f \in A\} .$$

Clearly $x_0 \in Y$ and since I is a proper ideal in A, $Y_0 = Y \backslash \{x_0\} \neq \emptyset$. Suppose x_0 is an isolated point of Y so that Y_0 is closed. Since X is completely regular, there exists

$g \in C(X)$ such that $g(x) = 1$ for all $x \in Y_0$ and $g(x_0) = 0$. Then $g + I = g|_Y$ is an identity in A/I . Suppose on the other hand that x_0 is a limit point of Y . Since Y is also a k-space, $C(Y)$ is complete ([24], Problem 8I) and so

$$\widehat{A/I} \subseteq \{f \in C(Y) : f(x_0) = 0\} \quad .$$

For each $y \in Y_0$, there exists an $f \in C(X)$ such that $f(y) = 1$ and $f(x_0) = 0$ from which it follows that if $\widehat{A/I}$ has an identity e , we must have $e(y) = 1$ for all $y \in Y\backslash\{x_0\}$ and $e(x_0) = 0$ which is impossible.

For a specific example we may take $X = [0, 1]$, $x_0 = 1$, $Y = [0, \frac{1}{2}] \cup \{1\}$ in the first case and $Y = [\frac{1}{2}, 1]$ in the second. In this example A_1 is B-complete as a locally convex space, being a Banach space. Then A/I is complete for any closed ideal I .

For an example where A/I need not be complete we may take X to be the (deleted) Tychonoff plank ([12], 8.20).

If ω_1 is the first uncountable ordinal, and if ω is the first infinite ordinal, then the Tychonoff plank is defined to be

$$T = \{\sigma : \sigma \leq \omega_1\} \times \{\sigma : \sigma \leq \omega\} \setminus \{(\omega_1, \omega)\} ,$$

where both ordinal spaces $\{\sigma : \sigma \leq \omega_1\}$ and $\{\sigma : \sigma \leq \omega\}$ are given the interval topology. Now

$$W = \{(\sigma, \omega) : \sigma < \omega_1\} ,$$
$$N = \{(\omega_1, n) : n \in \mathbb{N}\}$$

are disjoint closed subsets of T which are not completely separated. Therefore $Y = W \cup N$ is a closed subset of T which is not C-embedded. Thus, if $I = I_Y$ as above, then

$$C(T)/I = \{f|_Y : f \in C(T)\} \neq C(Y)$$

and hence $C(T)/I$ is a proper dense subspace of $C(Y)$. Take $x_0 = (\omega, \omega)$ which is a limit point of Y . $A/I = \{f|_Y : f \in A\}$ cannot be closed in $C(Y)$, for it is one-codimensional in $C(T)/I$ which would imply that $C(T)/I$ is closed in $C(Y)$. Therefore A/I is not complete.

§4. A new multiplication on $E \times \phi$

Let E be a locally convex algebra and let ϕ have the convolution multiplication

$$(\lambda_n) * (\mu_n) = \left(\sum_{r=1}^{n} \lambda_r \mu_{n-r+1}\right), \quad ((\lambda_n), (\mu_n) \in \phi) .$$

Then the operation . defined on $E \times \phi$ by

$$(x, (\lambda_n)).(y, (\mu_n)) = (xy + x\sum_{n=1}^{\infty} \mu_n + y\sum_{n=1}^{\infty} \lambda_n, (\lambda_n)*(\mu_n)),$$

$$(x, y \in E, (\lambda_n), (\mu_n) \in \phi)$$

is a jointly continuous multiplication on $E \times \phi$ making it into a locally convex algebra.

To show that this operation is a multiplication we only show that the associative law holds; the rest is clear. First note that if $(\gamma_n) = (\lambda_n)*(\mu_n)$, then

$$\sum_{n=1}^{\infty} \gamma_n = \sum_{n=1}^{\infty} \left(\sum_{r=1}^{n} \lambda_r \mu_{n-r+1}\right) = \left(\sum_{n=1}^{\infty} \lambda_n\right)\left(\sum_{n=1}^{\infty} \mu_n\right) .$$

Now for $(x, (\lambda_n)), (y, (\mu_n)), (z, (\nu_n)) \in E \times \phi$ we have

$$((x, (\lambda_n)).(y, (\mu_n))).(z, (\nu_n))$$

$$= (xy + x\sum_{n=1}^{\infty}\mu_n + y\sum_{n=1}^{\infty}\lambda_n, (\lambda_n)*(\mu_n)).(z, (\nu_n))$$

$$= ((xy + x\sum_{n=1}^{\infty}\mu_n + y\sum_{n=1}^{\infty}\lambda_n)z + (xy + x\sum_{n=1}^{\infty}\mu_n + y\sum_{n=1}^{\infty}\lambda_n)\sum_{n=1}^{\infty}\nu_n$$

$$+ z(\sum_{n=1}^{\infty}\lambda_n)(\sum_{n=1}^{\infty}\mu_n), ((\lambda_n)*(\mu_n))*(\nu_n))$$

$$= (x(yz + y\sum_{n=1}^{\infty}\nu_n + z\sum_{n=1}^{\infty}\mu_n) + x(\sum_{n=1}^{\infty}\mu_n)(\sum_{n=1}^{\infty}\nu_n)$$

$$+ (yz + y\sum_{n=1}^{\infty}\nu_n + z\sum_{n=1}^{\infty}\mu_n)\sum_{n=1}^{\infty}\lambda_n, (\lambda_n)*((\mu_n)*(\nu_n)))$$

$$= (x, (\lambda_n)).(yz + y\sum_{n=1}^{\infty}\nu_n + z\sum_{n=1}^{\infty}\mu_n, (\mu_n)*(\nu_n))$$

$$= (x, (\lambda_n)).((y, (\mu_n)).(z, (\nu_n))) .$$

Continuity of

$$((x, (\lambda_n)), (y, (\mu_n))) \to (xy + x\sum_{n=1}^{\infty}\mu_n + y\sum_{n=1}^{\infty}\lambda_n, (\lambda_n)*(\mu_n))$$

follows from the continuity of the algebraic operations on E and ϕ and the continuity of $(\lambda_n) \to \sum_{n=1}^{\infty}\lambda_n$ from ϕ onto $\mathbb{K}$.

This is a natural extension of unitization. Throughout this section ϕ will have convolution multiplication and on $E \times \phi$ we shall always assume the above mentioned multiplication which we will denote by . and will call dot multiplication.

With the convolution like multiplication on $\mathbb{K}^n$ defined as

$$(\alpha_i)*(\beta_i) = (\sum_{j=1}^{i}\alpha_j\beta_{i-j+1}), \quad ((\alpha_i), (\beta_i) \in \mathbb{K}^n)$$

we cannot define dot multiplication on $E \times \mathbb{K}^n$ for $n > 1$; the associative law fails, e.g. for $n = 2$, $x \in E$ we have

$$((0, (0, 1)).(0, (0, 1))).(x, (1, 1)) = (0, (0, 0)).\ (x, (1, 1))$$
$$= (0, (0, 0)) ,$$
$$(0, (0, 1)).((0, (0, 1)).(x, (1, 1))) = (0, (0, 1)).(x, (0, 1))$$
$$= (x, (0, 0)) .$$

For the case $n = 1$, $E \times \mathbb{K}$ with dot multiplication is just the unitization of E .

Similar to the $E \times \phi$ case, if L is a closed ideal in ϕ with $\sum_{n=1}^{\infty} \lambda_n = 0$ for all $(\lambda_n) \in L$, then the operation

$$(x, (\lambda_n) + L).(y, (\mu_n) + L) = (xy + x\sum_{n=1}^{\infty}\mu_n + y\sum_{n=1}^{\infty}\lambda_n, (\lambda_n)*(\mu_n) + L) ,$$

$$(x, y \in E, (\lambda_n), (\mu_n) \in \phi)$$

is a jointly continuous multiplication on $E \times (\phi/L)$ and $E \times (\phi/L)$ under this multiplication is a locally convex algebra. The condition $\sum_{n=1}^{\infty} \lambda_n = 0$ for all $(\lambda_n) \in L$ ensures that the multiplication is well-defined and the rest is straightforward.

The following Proposition shows a connection between the unitization of an algebra E and $E \times \phi$ with dot multiplication.

<u>4.1 Proposition</u>

The unitization E_1 of a locally convex algebra E is the quotient of $E \times \phi$ by the closed ideal $I = \{0\} \times L$ where $L = \{(\lambda_n) : \sum_{n=1}^{\infty} \lambda_n = 0\}$.

<u>Proof</u>

L is in fact the kernel of the continuous multiplicative linear functional

$$(\lambda_n) \to \sum_{n=1}^{\infty} \lambda_n$$

on ϕ. So L is a closed ideal of ϕ. Now I is clearly a closed subspace of $E \times \phi$ and for $(0, (\lambda_n)) \in I$, $(x, (\mu_n)) \in E \times \phi$ we have

$$\begin{aligned}(0, (\lambda_n)).(x, (\mu_n)) &= (x, (\mu_n)).(0, (\lambda_n)) \\ &= (x \sum_{n=1}^{\infty} \lambda_n, (\lambda_n)*(\mu_n)) \\ &= (0, (\lambda_n)*(\mu_n)) \in I .\end{aligned}$$

Thus I is a closed ideal of $E \times \phi$.

The mapping

$$h : (E \times \phi)/I \to E \times (\phi/L)$$

defined by

$$h((x, (\mu_n)) + I) = (x, (\mu_n) + L)$$

is obviously an algebraic isomorphism. If U and V are neighbourhoods of 0 in E and ϕ respectively, we have

$$h(U \times V + I) = U \times (V + L)$$

which ensures that h is a topological isomorphism. Also the mapping

$$g(x, (\mu_n) + L) = (x, \sum_{n=1}^{\infty} \mu_n)$$

is clearly a linear continuous open mapping of $E \times (\phi/L)$ onto E_1. If $(x, (\mu_n) + L) \neq 0$, then $x \neq 0$ or $(\mu_n) \notin L$ i.e. $x \neq 0$ or $\sum_{n=1}^{\infty} \mu_n \neq 0$, hence

$$g(x, (\mu_n) + L) = (x, \sum_{n=1}^{\infty} \mu_n) \neq 0 .$$

Finally

$$\begin{aligned} & g((x, (\mu_n) + L).(y, (\nu_n) + L)) \\ & = g(xy + x\sum_{n=1}^{\infty} \nu_n + y\sum_{n=1}^{\infty} \mu_n, (\mu_n)*(\nu_n) + L) \\ & = (xy + x\sum_{n=1}^{\infty} \nu_n + y\sum_{n=1}^{\infty} \mu_n, (\sum_{n=1}^{\infty} \mu_n)(\sum_{n=1}^{\infty} \nu_n)) \\ & = (x, \sum_{n=1}^{\infty} \mu_n)(y, \sum_{n=1}^{\infty} \nu_n) . \end{aligned}$$

Therefore g is also a topological isomorphism and hence we conclude that

$$(E \times \phi)/I \simeq E \times (\phi/L) \simeq E_1 . \qquad \Delta$$

Now we look at B- and B_r-completeness of $E \times \phi$ if E is so. As the following Theorem shows, B_r-completeness of $E \times \phi$ follows from B_r-completeness of E ; but this is not the case for B-complete algebras by Example 4.7.

4.2 Theorem

If E is a B_r-complete algebra so also is $E \times \phi$.

Proof

Suppose E is also complete. Let t be a continuous one-to-one nearly open algebra homomorphism of $E \times \phi$ onto a locally convex algebra F . $t|_E$ is nearly open (Lemma 1.3(a)), hence open onto $t(E)$. Therefore $t(E)$ being complete is closed in F . It follows then by Lemma 1.3(b) that t is open, so $E \times \phi$ is B_r-complete.

Now suppose E is not complete. $\hat{E} \times \phi$ is then B_r-complete by Theorem 1.2(a) and the first part of the proof. Let I be a non-zero closed ideal of $\hat{E} \times \phi$. Suppose there is $(x, (\lambda_n)) \in I$ with $(\lambda_n) \neq 0$. If $k \in \mathbb{N}$ is the greatest number such that $\lambda_k \neq 0$, and if (μ_n) is the member of ϕ defined as, $\mu_1 = 1$, $\mu_{k+1} = -1$ and $\mu_n = 0$ for $n \neq 1, k+1$, then the kth component of $(\lambda_n)*(\mu_n)$ is $\lambda_k \neq 0$. Hence $(x, (\lambda_n)).(0, (\mu_n)) = (0, (\lambda_n)*(\mu_n))$ is a non-zero element of $I \cap (E \times \phi)$. If $(\lambda_n) = 0$ for all $(x, (\lambda_n)) \in I$, then $J = \{x : (x, 0) \in I\}$ is a non-zero closed ideal of $\hat{E}$. Thus by Theorem 1.2(a), $J \cap E \neq \{0\}$ and therefore $I \cap (E \times \phi) \neq \{0\}$. In each case we conclude from Theorem 1.2(a) that $E \times \phi$ is B_r-complete. Δ

4.3 Remark

If E is B_r-complete, E_1 may fail to be B_r-complete (Example 3.2) although $E \times \phi$ is always B_r-complete. Using Proposition 4.1 we get examples of B_r-complete algebras having quotient algebras which are not B_r-complete.

In the course of proving B_r-completeness of $E \times \phi$ we use a one-to-one homomorphism (which is also continuous and nearly open) of $E \times \phi$ onto a locally convex algebra F. The following Proposition shows that we only need to verify that the homomorphism is one-to-one on E and on ϕ.

4.4 Proposition

Let t be an algebra homomorphism of $E \times \phi$ into an algebra F such that $t|_E$ and $t|_\phi$ are one-to-one. Then t is one-to-one.

Proof

Let $(x, (\lambda_n)) \in \ker t$. $\forall\ (y, (\mu_n)) \in E \times \phi$ we have

$$(x, (\lambda_n)).(y, (\mu_n)) = (xy + x\sum_{n=1}^{\infty}\mu_n + y\sum_{n=1}^{\infty}\lambda_n,\ (\lambda_n)*(\mu_n)) \in \ker t\ . \qquad (*)$$

Put $(\mu_n) = 0$ in (*) to get

$$(xy + y\sum_{n=1}^{\infty}\lambda_n,\ 0) \in \ker t \qquad \forall\ y \in E\ .$$

And put $y = 0$, $\sum_{n=1}^{\infty}\mu_n = 0$ to get

$$(0,\ (\lambda_n)*(\mu_n)) \in \ker t \qquad \forall\ (\mu_n) \in \phi \text{ with } \sum_{n=1}^{\infty}\mu_n = 0\ .$$

Since t is one-to-one on both E and ϕ we must then have

$$\text{(i)} \quad xy + y\sum_{n=1}^{\infty}\lambda_n = 0 \qquad \forall\ y \in E$$

$$\text{(ii)} \quad (\lambda_n)*(\mu_n) = 0 \qquad \forall\ (\mu_n) \in \phi \text{ with } \sum_{n=1}^{\infty}\mu_n = 0\ .$$

If we put $(\mu_n) = (1, -1, 0, 0, \ldots)$ in (ii) we get

$(\lambda_n)*(1, -1, 0, 0, \ldots) = (\lambda_1, \lambda_2 - \lambda_1, \lambda_3 - \lambda_2, \ldots) = 0$, which implies that $(\lambda_n) = 0$. Now from (i), $xy = 0 \ \forall\ y \in E$, and if in (*) we put (μ_n) such that $\sum_{n=1}^{\infty} \mu_n \neq 0$ and note that $t|_E$ is one-to-one we must have $x \sum_{n=1}^{\infty} \mu_n = 0$, from which we get $x = 0$. Hence $(x, (\lambda_n)) = 0$ and therefore t is one-to-one. Δ

4.5 Lemma

Let E be a locally convex algebra. Let I be a non-empty subset of E and let $J = \{(x, 0) : x \in I\} \subseteq E \times \phi$. Then I is a closed ideal of E if and only if J is a closed ideal of $E \times \phi$. In this case $(E \times \phi)/J$ is topologically isomorphic with $(E/I) \times \phi$.

Proof

It is clear that I is a closed subspace of E if and only if J is a closed subspace of $E \times \phi$. If I is an ideal of E , then for $(x, 0) \in J$ and $(y, (\mu_n)) \in E \times \phi$ we have $x \in I$, hence

$$(x, 0).(y, (\mu_n)) = (xy + x \sum_{n=1}^{\infty} \mu_n, 0) = (xy, 0) + (x \sum_{n=1}^{\infty} \mu_n, 0) \in J ,$$

$$(y, (\mu_n)).(x, 0) = (yx + x \sum_{n=1}^{\infty} \mu_n, 0) = (yx, 0) + (x \sum_{n=1}^{\infty} \mu_n, 0) \in J ,$$

which implies that J is an ideal of $E \times \phi$. Now let J be an ideal of $E \times \phi$ and let $x \in I, y \in E$. We have $(x, 0) \in J$ and $(y, 0) \in E \times \phi$; hence

$$(x, 0).(y, 0) = (xy, 0) \in J \implies xy \in I ,$$

$$(y, 0).(x, 0) = (yx, 0) \in J \implies yx \in I .$$

Therefore I is an ideal of E .

For the last part, define $h : (E \times \phi)/J \to (E/I) \times \phi$ by $h((x, (\lambda_n)) + J) = (x + I, (\lambda_n))$. h is well-defined, for if $(x, (\lambda_n)) + J = (y, (\mu_n)) + J$, then $(x - y, (\lambda_n) - (\mu_n)) \in J$ i.e. $(\lambda_n) = (\mu_n)$ and $x - y \in I$. Hence $(x + I, (\lambda_n)) = (y + I, (\mu_n))$. It is easily seen that h is an algebraic isomorphism. It is also a topological isomorphism. For, if U is a basic neighbourhood of 0 in E and V is a basic neighbourhood of 0 in ϕ, then $(U + I) \times V$, (resp. $U \times V + J$) is a basic neighbourhood of 0 in $(E/I) \times \phi$ (resp. $(E \times \phi)/J$) and we have

$$\begin{aligned} h(U \times V + J) &= \{h((x, (\lambda_n)) + J : x \in U, (\lambda_n) \in V\} \\ &= \{(x + I, (\lambda_n)) : x \in U, (\lambda_n) \in V\} \\ &= (U + I) \times V . \end{aligned}$$

Δ

<u>4.6 Theorem</u>

Let E be a B-complete algebra. Then $E \times \phi$ is B-complete if and only if E_1 is B-complete.

<u>Proof</u>

Since, by Proposition 4.1, E_1 is a quotient of $E \times \phi$, if $E \times \phi$ is B-complete, so also is E_1.

Suppose E_1 is B-complete and let t be a continuous nearly open algebra homomorphism of $E \times \phi$ onto a locally convex algebra F. Clearly $J = \{(x, 0) : t(x, 0) = 0\}$ is a closed ideal in $E \times \phi$. Then by Lemma 4.5, $I = \{x : (x, 0) \in J\}$ is a closed ideal in E and we have

$$(E \times \phi)/J \cong (E/I) \times \phi .$$

Define $s : (E/I) \times \phi \to F$ by $s(x + I, (\lambda_n)) = t(x, (\lambda_n))$. Clearly s is a well-defined, continuous nearly open algebra homomorphism of $(E/I) \times \phi$ onto F which is one-to-one on E/I . If we show that s is open it will follow that t is open and the proof will be complete. But since the quotient of a B-complete algebra by a closed ideal is B-complete, and because of Theorem 3.6 it is therefore enough to consider the situation where $t|_E$ is one-to-one and either $\hat{E}$ has no identity or E has an identity.

In this case let $\hat{t} : \hat{E} \times \phi \to G$ be the extension of t by continuity mapping $\hat{E} \times \phi$ onto a subalgebra G of $\hat{F}$. We show that $\ker t = \ker \hat{t}$. Let $(x, (\lambda_n)) \in \ker \hat{t}$. We have

$$\hat{t}((x,(\lambda_n)).(y, 0)) = \hat{t}(x, (\lambda_n))\hat{t}(y, 0) = 0 \qquad \forall\ y \in \hat{E}$$

and $\hat{t}((y, 0).(x, (\lambda_n))) = \hat{t}(y, 0)\hat{t}(x, (\lambda_n)) = 0 \qquad \forall\ y \in \hat{E}$.

Hence

$$\hat{t}(xy + y\sum_{n=1}^{\infty}\lambda_n, 0) = 0 = \hat{t}(yx + y\sum_{n=1}^{\infty}\lambda_n, 0) \qquad \forall\ y \in \hat{E} .$$

Since, as before, $t|_E$ is a topological isomorphism onto $t(E)$ its extension by continuity to $\hat{E}$, i.e. $\widehat{t|_E} = \hat{t}|_{\hat{E}}$ is also a topological isomorphism onto $\hat{t}(\hat{E})$ ([37], Ch. VI, Proposition 5, Corollary 1). Therefore

$$xy + y\sum_{n=1}^{\infty}\lambda_n = 0 = yx + y\sum_{n=1}^{\infty}\lambda_n \qquad \forall\ y \in \hat{E} .$$

If $\sum_{n=1}^{\infty}\lambda_n \neq 0$ we get immediately that $(-1/\sum_{n=1}^{\infty}\lambda_n)x$ is an identity in $\hat{E}$. Then E must have an identity by assumption and this identity must be $(-1/\sum_{n=1}^{\infty}\lambda_n)x$ i.e. $x \in E$ and so $(x, (\lambda_n)) \in E \times \phi$. If $\sum_{n=1}^{\infty}\lambda_n = 0$ we have $xy = yx = 0\ \forall\ y \in \hat{E}$.

Then by Lemma 2.1, $x \in E$ and again $(x, (\lambda_n)) \in E \times \phi$. Therefore $(x, (\lambda_n)) \in \ker t$, hence $\ker t = \ker \hat{t}$.

Now as in the last paragraph of the proof of Lemma 2.2 we conclude that $t(E)$ is closed in F and hence by Lemma 1.3(b), t is open. Δ

4.7 Example

In Example 3.2 we showed that $\mathcal{K}$, the algebra of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ with compact support, is a B-complete algebra, but $\mathcal{K}_1$ is not even B_r-complete. By Proposition 4.1, $\mathcal{K}_1$ is topologically isomorphic with $(\mathcal{K} \times \phi)/I$, where $I = \{0\} \times L$ is a closed ideal of $\mathcal{K} \times \phi$ with $L = \{(\lambda_n) \in \phi : \sum_{n=1}^{\infty} \lambda_n = 0\}$. Hence $(\mathcal{K} \times \phi)/I$ is not B-complete which implies that $\mathcal{K} \times \phi$ is not B-complete although it is B_r-complete.

Remark

Note that if X is a k-space and if $C^*(X)$ is the subalgebra of $C(X)$ consisting of all bounded functions, then $C(X)$ and $C^*(X)$ are both B-complete ([38], Theorem 3.2 and Corollary 2.5). Hence by Corollary 3.7 and Theorem 4.6, $C(X) \times \phi$ and $C^*(X) \times \phi$ are also B-complete. But, given a k-space X , there is no k-space Y such that $C(X) \times \phi = C(Y)$. For, if there is such a Y , then ϕ would be a quotient of $C(Y)$ and therefore ϕ would be $C(Z)$ for some closed subspace Z of Y . This follows because by Rosa ([38], Proof of Theorem 3.2) there exists a closed subspace Z of Y such that ϕ is a dense subspace of $C(Z)$ and since ϕ is complete we must have $\phi = C(Z)$. Now if Z has an infinite compact subset

W , then $C(Z)$ must have dimension $\geq c$, c the cardinality of the real line; for $C(W)$ is an infinite dimensional Banach space, therefore $\dim C(W) \geq c$, and W is C-embedded in X ([51], Corollary 8.5.1). Since the dimension of ϕ is $\aleph_0$ this is impossible. So the compact subsets of Z are finite. But then the equicontinuous subsets of the dual of $C(Z)$ should be finite dimensional. In this case $C(Z)$ cannot have its finest locally convex topology, since it is infinite dimensional.

§5. Some related topics

5.1 Multiplications on ϕ

In the introductory section of the present chapter we mentioned that for a "suitable" multiplication on ϕ it is possible to define a multiplication on $E \times \phi$ for an algebra E . In the beginning of the Section 4 we defined dot multiplication on $E \times \phi$ when ϕ had convolution multiplication and we pointed out that it does not define a multiplication on $E \times \mathbb{K}^n$ for $n > 1$ when $\mathbb{K}^n$ has a convolution like multiplication defined earlier. It is easy to see that we do not get a multiplication if we replace convolution by coordinatewise multiplication in the definition of dot multiplication on $E \times \phi$, e.g. for $x \in E$ we have

$$((x, (1, 0, 0, \ldots)).(0, (1, 0, 0, \ldots))).(0, (1, 1, 0, 0, \ldots)) = (2x, (1, 0, 0, \ldots)) ,$$

but

$$(x, (1, 0, 0, \ldots)).((0, (1, 0, 0, \ldots)).(0, (1, 1, 0, 0, \ldots))) = (x, (1, 0, 0, \ldots)) ,$$

so the associative law fails. Also in the proof of Theorem 4.2 we made specific use of the type of multiplication on ϕ to establish the B_r-completeness of $E \times \phi$. These observations suggest some questions as follows.

(a) What kind of multiplication on ϕ enables us to define a corresponding . multiplication?

(b) When is $(E \times \mathbb{K}^n, .)$ an algebra?

(c) What are necessary conditions on the multiplication of ϕ to make $E \times \phi$ into a B_r-complete algebra when E is so?

Here we want to consider these questions.

As we have mentioned earlier, the only condition on the multiplication of ϕ we need to be able to define dot multiplication on $E \times \phi$, is that for $(\lambda_n), (\mu_n) \in \phi$ if $(\gamma_n) = (\lambda_n)(\mu_n)$, then we must have

$$\sum_{n=1}^{\infty} \gamma_n = (\sum_{n=1}^{\infty} \lambda_n)(\sum_{n=1}^{\infty} \mu_n) .$$

This occurs if we regard ϕ as a semigroup ring of a countable semigroup over the field $\mathbb{K}$. For if m is a semigroup operation on the countable set $S = (s_1, s_2, s_3, \ldots)$ then the elements of the semigroup ring $\mathbb{K}S$ are of the form

$$\sum_{n=1}^{\infty} \lambda_n s_n , \lambda_n \in \mathbb{K} , \quad s_n \in S$$

with $\lambda_n = 0$ except for finitely many n. The multiplication on $\mathbb{K}S$ is then of the form

$$(\sum_{n=1}^{\infty} \lambda_n s_n)(\sum_{n=1}^{\infty} \mu_n s_n) = \sum_{n=1}^{\infty} (\sum_{m(i,j)=n} \lambda_i \mu_j) s_n .$$

Putting $e_1 = (1, 0, 0, \ldots)$, $e_2 = (0, 1, 0, 0, \ldots)$, ... we can write each element (λ_n) of ϕ as $\sum_{n=1}^{\infty} \lambda_n e_n$. Then with the above observation

$$(\lambda_n)(\mu_n) = (\sum_{n=1}^{\infty} \lambda_n e_n)(\sum_{n=1}^{\infty} \mu_n e_n) = \sum_{n=1}^{\infty} (\sum_{m(i,j)=n} \lambda_i \mu_j) e_n$$

$$= (\sum_{m(i,j)=n} \lambda_i \mu_j) .$$

Without loss of generality we can take $\mathbb{N}$ as our countable set with a function $m : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with the property $m(m(i,j), k) = m(i, m(j, k))$ for all $i, j, k \in \mathbb{N}$. Then the function $M : \phi \times \phi \to \phi$ defined by $M((\lambda_n), (\mu_n)) = (\gamma_n)$ where $\gamma_n = \sum_{m(i,j)=n} \lambda_i \mu_j$ defines a multiplication on ϕ and we have

$$\sum_{n=1}^{\infty} \gamma_n = (\sum_{n=1}^{\infty} \lambda_n)(\sum_{n=1}^{\infty} \mu_n) .$$

We shall give a list of some such multiplications on ϕ later.

The reason for the fact that . is not a multiplication on $E \times \mathbb{K}^n$ for $n > 1$ when $\mathbb{K}^n$ has convolution like multiplication, is that with this multiplication $\mathbb{K}^n$ is not a subalgebra of ϕ . If it happens that with respect to a semigroup ring multiplication on ϕ , $\mathbb{K}^n$ is a subalgebra of ϕ then $E \times \mathbb{K}^n$ would also be an algebra under dot multiplication. In the following table we have shown which multiplications have this property; the proofs are straightforward.

It is easily seen from the proof of Theorem 4.2 that if we have any multiplication on ϕ , instead of convolution, for which we could define dot multiplication on $E \times \phi$, then B_r-completeness of $E \times \phi$ will follow from the B_r-completness of E if the multiplication of ϕ has the property that for every non-zero element (λ_n) of ϕ there exists an element (μ_n) in ϕ such that $\sum_{n=1}^{\infty} \mu_n = 0$ with $(\mu_n)(\lambda_n) \neq 0$ or $(\lambda_n)(\mu_n) \neq 0$. In this case we say that the multiplication has property (A) . Example 5.2 shows that there are multiplications on ϕ for which B_r-completeness of E does not imply B_r-completeness of $E \times \phi$.

To see that multiplications (3), (4), (6) and (7) in the table have property (A) , just proceed as in the last paragraph of the proof of Theorem 4.2 . For the multiplications (5), (8) and (9) in the table consider $(\lambda_n) = (1, 0, 0, \ldots)$. Then in each case $(\mu_n)(\lambda_n) = (\lambda_n)(\mu_n) = (\sum_{n=1}^{\infty} \mu_n)(1, 0, 0, \ldots)$ for all $(\mu_n) \in \phi$, and so if $\sum_{n=1}^{\infty} \mu_n = 0$, we have $(\lambda_n)(\mu_n) = (\mu_n)(\lambda_n) = 0$. For the case (1) if we take $(\lambda_n) = (1, -1, 0, 0, \ldots)$, then $\forall\, (\mu_n) \in \phi$ with $\sum_{n=1}^{\infty} \mu_n = 0$ we have $(\lambda_n)(\mu_n) = \sum_{n=1}^{\infty} \mu_n(\lambda_n) = 0$ and $(\mu_n)(\lambda_n) = \sum_{n=1}^{\infty} \lambda_n(\mu_n) = 0$. Multiplication (2) is similar to (1). Finally for (10) taking $(\lambda_n) = (1, 1, 0, 0, \ldots)$, for each $(\mu_n) \in \phi$ with $\sum_{n=1}^{\infty} \mu_n = 0$ we have

$$(\lambda_n)(\mu_n) = (\mu_n)(\lambda_n) = \sum_{n=1}^{\infty} \mu_n(\lambda_n) = 0 .$$

Therefore the multiplications (1), (2), (5), (8), (9) and (10) do not have property (A) .

	$m : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$	$M : \phi \times \phi \to \phi$ $M((\lambda_n), (\mu_n)) =$	Is $\mathbb{K}^n$ a subalgebra of ϕ ?	Has M property (A) ?
1	$m(i,j) = i$	$(\sum_{j=1}^{\infty} \lambda_n \mu_j)$	Yes	No
2	$m(i,j) = j$	$(\sum_{i=1}^{\infty} \lambda_i \mu_n)$	Yes	No
3	$m(i,j) = i + j - 1$ (*)	$(\sum_{i+j-1=n} \lambda_i \mu_j)$	No	Yes
4	$m(i,j) = ij$ (*)	$(\sum_{ij=n} \lambda_i \mu_j)$	No	Yes
5	$m(i,j) = i \wedge j$ (†)	$(\sum_{i \wedge j=n} \lambda_i \mu_j)$	Yes	No
6	$m(i,j) = i \vee j$ (†)	$(\sum_{i \vee j=n} \lambda_i \mu_j)$	Yes	Yes
7	$m(i,j) = [i,j]$ (†)	$(\sum_{[i,j]=n} \lambda_i \mu_j)$	No	Yes
8	$m(i,j) = (i,j)$ (†)	$(\sum_{(i,j)=n} \lambda_i \mu_j)$	Yes	No
9	$m(i,j) = \begin{cases} 1 & \text{if } i \wedge j = 1 \\ i \vee j & \text{otherwise} \end{cases}$	$(\sum_{i \wedge j=1} \lambda_i \mu_j, \sum_{i \vee j=2, \; i,j \neq 1} \lambda_i \mu_j, \ldots)$	Yes	No
10	$m(i,j) = \begin{cases} 1 & \text{if } i+j \text{ is odd} \\ 2 & \text{if } i+j \text{ is even} \end{cases}$		No if $n=1$ Yes if $n>1$	No

(*) The multiplication (3) is just convolution, and (4) is Dirichlet multiplication.
(†) $i \wedge j = \min\{i,j\}$, $i \vee j = \max\{i,j\}$, $[i,j]$ = least common multiple of i and j, (i,j) = greatest common divisor of i and j.

5.2 Example

Let E be a B_r-complete algebra with no identity such that $\hat{E}$ has an identity (take e.g. $E = \mathcal{K}$ of Example 3.2). Since $\hat{E}$ is complete, as in the first part of the proof of Theorem 4.2, we get that $\hat{E} \times \phi$ is B_r-complete (no matter what the multiplication of ϕ). But $E \times \phi$ is not B_r-complete if the multiplication of ϕ is for example number (5) in the table and $E \times \phi$ has the corresponding dot multiplication. For, then $I = \{(\lambda e, (-\lambda, 0, 0, \ldots)) : \lambda \in \mathbb{K}\}$, e the identity element of $\hat{E}$, is a non-zero closed ideal in $\hat{E} \times \phi$ with $I \cap (E \times \phi) = (0, 0)$, which implies by Theorem 1.2(a) that $E \times \phi$ cannot be B_r-complete.

Here we give another example which we promised earlier in the beginning of this chapter. This is an example of an extension of the multiplication of a locally convex algebra E to $E \times \phi$ which is not even separately continuous.

5.3 Example

Let $E = \phi[\sigma(\phi, \phi)]$. We show that E, with convolution multiplication, is a locally convex algebra. Let

$$(\lambda_n^{(m)}) \to 0, \quad (\mu_n^{(m)}) \to 0 \quad \text{in} \quad \omega \quad \text{as} \quad m \to \infty .$$

Then since evaluations are continuous,

$$\lambda_n^{(m)} \to 0, \; \mu_n^{(m)} \to 0 \quad \text{as} \quad m \to \infty \quad \text{for each} \quad n \in \mathbb{N} .$$

Therefore, for each n,

$$\sum_{r=1}^{n} \lambda_r^{(m)} \mu_{n-r+1}^{(m)} \to 0 \quad \text{as} \quad m \to \infty$$

and hence

$$(\lambda_n^{(m)})*(\mu_n^{(m)}) \to 0 \text{ in } \omega \ .$$

Thus convolution is a jointly continuous multiplication on ω . ω being a Fréchet space is B-complete and E is a dense subalgebra of it. Let I be a non-zero ideal in ω . Let (ν_n) be a non-zero element of I and let n_0 be the least n such that $\nu_n \neq 0$. For each $k \in \mathbb{N}$, the element $(\xi_n^{(k)})$ of ω is uniquely determined by the equations:

$$\nu_{n_0}\xi_1 = \nu_{n_0}$$

$$\nu_{n_0}\xi_2 + \nu_{n_0+1}\xi_1 = \nu_{n_0+1}$$

$$\dots\dots\dots\dots$$

$$\nu_{n_0}\xi_{k+1} + \nu_{n_0+1}\xi_k + \dots + \nu_{n_0+k}\xi_1 = \nu_{n_0+k}$$

$$\nu_{n_0}\xi_{k+2} + \nu_{n_0+1}\xi_{k+1} + \dots + \nu_{n_0+k}\xi_1 = 0$$

$$\dots\dots\dots\dots\dots\dots$$

Then

$$(\xi_n^{(k)})*(\nu_n) = (\nu_1, \nu_2, \dots, \nu_{n_0+k}, 0, 0, \dots)$$

$$\in I \cap E$$

Since k is arbitrary, it follows that $I \cap E$ is dense in I . Thus $E = \phi[\sigma(\phi, \phi)]$ is B-complete by Theorem 1.2.

Now as in §4 the operation

$$((\lambda_n), (\mu_n))((\lambda'_n), (\mu'_n))$$

$$= ((\lambda_n)*(\lambda'_n), (\mu_n)*(\mu'_n) + \sum_{n=1}^{\infty}\lambda_n(\mu'_n) + \sum_{n=1}^{\infty}\lambda'_n(\mu_n))$$

is a multiplication on $E \times \phi$ extending the multiplication of E .

However, it is not separately continuous. For, if e.g. $(\lambda_n^{(m)})$ is the member of E whose first m components are all $\frac{1}{m}$ and the rest zero, then $(\lambda_n^{(m)}) \to 0$ in E as $m \to \infty$. But if $(\mu_n) = (1, 0, 0, \dots)$, we have for all $m \in \mathbb{N}$.

$$((\lambda_n^{(m)}), 0)(0, (\mu_n)) = (0, \sum_{n=1}^{\infty} \lambda_n^{(m)} (\mu_n))$$

$$= (0, (\mu_n)) \not\to 0 \quad \text{as} \quad m \to \infty .$$

5.4 Definitions ([35], Section 2)

Let E and F be topological vector spaces and t a linear mapping of E into F. It is said that the filter condition holds with respect to t if, for each Cauchy filter base $\mathcal{F}$ on E such that $t(\mathcal{F})$ converges to a point of $t(E)$, then $\mathcal{F}$ converges to a point of E.

W. Robertson ([35], Theorem 1) showed that if E and F are Hausdorff topological vector spaces and if t is a continuous linear mapping of E into F then the filter condition holds if and only if $\ker t = \ker \hat{t}$, where $\hat{t}$ is the extension of t by continuity from the completion $\hat{E}$ of E into the completion $\hat{F}$ of F. We have used the equality $\ker t = \ker \hat{t}$ in the proofs of Lemma 2.2 and Theorem 4.6. This suggests a relation between the filter condition and B_r-completeness (B-completeness) of locally convex algebras which we have put in the following Theorem.

Let E be a locally convex algebra and let $\Phi = \phi$ or $\mathbb{K}^n$, as in Section 1, with its finest locally convex topology. Suppose $E \times \Phi$ has a jointly continuous multiplication which induces the given multiplication on E .

5.5 Theorem

Let E be a B_r-complete algebra. Then $E \times \Phi$ is B_r-complete if and only if the filter condition holds with respect to every one-to-one, continuous nearly open algebra homomorphism t of $E \times \Phi$ onto an arbitrary locally convex algebra F .

Proof

If $E \times \Phi$ is B_r-complete such a mapping t should be open and hence by Proposition 1 of [35] the filter condition holds.

For the converse, let t be a continuous one-to-one nearly open algebra homomorphism of $E \times \Phi$ onto a locally convex algebra F and suppose the filter condition holds. The extension $\hat{t}$ of t by continuity is clearly a continuous nearly open algebra homomorphism of $\hat{E} \times \Phi$ onto $\hat{t}(\hat{E} \times \Phi) \subseteq \hat{F}$ which is also one-to-one by Theorem 1 of [35]. As in the first part of the proof of Lemma 2.2 it follows that $\hat{t}(\hat{E})$ is closed in $\hat{t}(\hat{E} \times \Phi)$. Then $t(E) = \hat{t}(\hat{E}) \cap F$ is closed in F and hence t is open by Lemma 1.3(b). Δ

5.6 Definitions

Let E be a locally convex algebra. A subset B of E is called idempotent if $xy \in B$ for all $x, y \in B$. An m-barrel in E is a barrel which is also idempotent. E is called an m-barrelled algebra if each m-barrel in E is a neighbourhood of the origin. A topological algebra E is called locally m-convex if there exists a base of absolutely convex idempotent neighbourhoods of 0 .

5.7 Theorem

A locally convex algebra E is m-barrelled if and only if whenever there is an algebra homomorphism t of E onto a dense subalgebra of a locally m-convex B_r-complete algebra F whose graph is closed, then t is continuous.

Proof

In view of Theorem 5.1 of [38] for the if part it is sufficient to show that for each neighbourhood U of 0 in F, $\mathrm{cl}\ t^{-1}(U)$ is a neighbourhood of 0 in E. Now since F is locally m-convex, given such a U, there is an absolutely convex neighbourhood V of 0 in F such that $VV \subseteq V \subseteq U$. $\mathrm{cl}\ t^{-1}(V)$ is clearly a barrel in E. Let $x, y \in t^{-1}(V)$; then $t(xy) = t(x)t(y) \in VV \subseteq V$, i.e. $xy \in t^{-1}(V)$. It follows that $t^{-1}(V)$ and hence $\mathrm{cl}\ t^{-1}(V)$ are idempotent, for

$$\mathrm{cl}\ t^{-1}(V)\ \mathrm{cl}\ t^{-1}(V) \subseteq \mathrm{cl}(t^{-1}(V)t^{-1}(V)) \subseteq \mathrm{cl}\ t^{-1}(V) .$$

Therefore $\mathrm{cl}\ t^{-1}(V)$ is a neighbourhood of 0 in E and we have $\mathrm{cl}\ t^{-1}(V) \subseteq \mathrm{cl}\ t^{-1}(U)$.

For the converse let B be an m-barrel in E. Put $N = \bigcap_{\lambda>0}\lambda B$. N is an ideal in E; for if $x \in N$ and $y \in E$ then $x \in \lambda B$ for all $\lambda > 0$ and there exists $\mu > 0$ such that $y \in \mu B$. Hence $xy \in (\lambda B)(\mu B) \subseteq \lambda\mu B\ \forall\ \lambda > 0$ which implies that $xy \in \bigcap_{\lambda>0}\lambda\mu B = N$. Similarly $yx \in N$. The result follows as in [37], Ch. VI, Proposition 11 if we show that the norm on E/N defined by B is in fact an algebra norm. For this let $x, y \in E$; there are $\lambda, \mu > 0$ such that $x \in \lambda B$, $y \in \mu B$. Thus $xy \in \lambda\mu BB \subseteq \lambda\mu B$, which implies that

$$\{\nu > 0 : x \in \nu B\}\{\nu > 0 : y \in \nu B\} \subseteq \{\nu > 0 : xy \in \nu B\} \quad .$$

Hence

$$\|xy + N\| \leq \|x + N\| \; \|y + N\| \quad . \qquad \Delta$$

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