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ON GRAPHS WITH AN EIGENVALUE OF MAXIMAL MULTIPLICITY

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Abstract
Let \( G \) be a graph of order \( n \) with an eigenvalue \( \mu \neq -1, 0 \) of multiplicity \( k < n - 2 \). It is known that \( k \leq n + \frac{1}{2} - \sqrt{2n + \frac{1}{4}} \), equivalently \( k \leq \frac{1}{2}t(t-1) \), where \( t = n - k > 2 \). The only known examples with \( k = \frac{1}{2}t(t-1) \) are \( 3K_2 \) (with \( n = 6, \mu = 1, k = 3 \)) and the maximal exceptional graph \( G_{36} \) (with \( n = 36, \mu = -2, k = 28 \)). We show that no other example can be constructed from a strongly regular graph in the same way as \( G_{36} \) is constructed from the line graph \( L(K_9) \).

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1 Introduction

Let $G$ be a graph of order $n$ with an eigenvalue $\mu \neq -1, 0$ of multiplicity $k < n - 2$. It was shown in [1] that $k \leq n + \frac{1}{2} - \sqrt{2n + \frac{1}{4}}$, equivalently $k \leq \frac{1}{2}t(t-1)$, where $t = n - k > 2$. The only known examples with $k = \frac{1}{2}t(t-1) > 1$ are $3K_2$, with spectrum $-1^{(3)}, 1^{(3)}$, and the unique maximal exceptional graph of order 36, with spectrum $21, 5^{(7)}, -2^{(28)}$. The latter graph is described in [3, Chapter 6] and [4, Example 5.2.6(a)]; it is denoted here by $G_{36}$. After a decade, it remains a problem to determine all the graphs with $k = \frac{1}{2}t(t-1)$. The restricted question, of similar standing, is whether further examples can be constructed from a strongly regular graph in the same way that $G_{36}$ is constructed from the line graph $L(K_9)$. Here we answer this question in the negative.

To describe the construction we recall some notation and terminology from [4]. For a subset $X$ of the vertex set $V(G)$, we write $X$ for $V(G) \setminus X$, $G - X$ for the subgraph of $G$ induced by $X$, and $G_X$ for the graph obtained from $G$ by switching with respect to $X$. We say that $X$ is a star set for $\mu$ if $|X| = k$ and $\mu$ is not an eigenvalue of $G - X$. Our main result is the following.

**Theorem 1.1.** Let $G$ be a graph of order $\frac{1}{2}t(t+1)$ ($t > 2$) with an eigenvalue $\mu \notin \{-1, 0\}$ of multiplicity $\frac{1}{2}t(t-1)$. Suppose that $G$ has a star set $X$ for $\mu$ such that (i) $X \cup \overline{X}$ is an equitable partition of $G$, (ii) $G_X$ is a strongly regular graph. Then $t = 8$, $\mu = -2$ and $G = G_{36}$.

Note that, in the situation of Theorem 1.1, $X \cup \overline{X}$ is also an equitable partition of $G_X$. To construct $G_{36}$, we take $G_X = L(K_9)$ and choose $X$ so that $X$ induces $L(K_8)$ and $\overline{X}$ induces $K_8$.

2 Prerequisites

If $X$ is a star set for $\mu$ in $G$, then $G - X$ is said to be a star complement for $\mu$ in $G$. Star sets and star complements exist for any eigenvalue of any graph, and their basic properties are described in [4, Chapter 5]. In particular, we shall require the following result.

**Theorem 1.1** [4, Theorem 5.1.7] Let $X$ be a set of $k$ vertices in the graph $G$ and suppose that $G$ has adjacency matrix $A = \begin{pmatrix} A_X & B^T \\ B & C \end{pmatrix}$, where $A_X$ is the adjacency matrix of the subgraph induced by $X$. Then $X$ is a star set for $\mu$ in $G$ if and only if $\mu$ is not an eigenvalue of $C$ and

$$\mu I - A_X = B^T (\mu I - C)^{-1} B. \quad (1)$$
Suppose that $X$ is a star set, and let $H = G - X$. In Theorem 1.1, $k$ is the multiplicity of $\mu$, and $C$ is the adjacency matrix of $H$. Also, the columns of $B$ are the characteristic vectors of the $H$-neighbourhoods

$$\Delta_H(u) = \{ v \in V(H) : u \sim v \} \ (u \in X),$$

where we write ‘$u \sim v$’ to mean that vertices $u, v$ are adjacent in $G$. Equation (1) shows that any graph is determined by an eigenvalue $\mu$, a star complement $H = G - X$ and the $H$-neighbourhoods of vertices in $X$. When $G - X$ is complete, we obtain the following by equating diagonal entries in Equation (1).

**Lemma 2.2.** Suppose that $X$ is a star set for $\mu$ in the graph $G$. Let $H = G - X$, $u \in X$. If $H = K_t \ (t > 2)$ and $|\Delta_H(u)| = a$ then

$$a^2 - (t - \mu - 1)a + \mu(\mu + 1)(t - \mu - 1) = 0.$$

In the general case, we let $|V(H)| = t > 2$ and define a bilinear form on $\mathbb{R}^t$ by

$$\langle \langle x, y \rangle \rangle = x^\top (\mu I - C)^{-1} y \ (x, y \in \mathbb{R}^t).$$

We let $V(G) = \{1, 2, \ldots, n\}$ and write $S = (B|C - \mu I)$, with columns $s_u \ (u = 1, \ldots, n)$. Let $Q_t$ denote the space of homogeneous quadratic functions on $\mathbb{R}^t$. We define $F_1, \ldots, F_n \in Q_t$ by

$$F_u(x) = \langle \langle s_u, x \rangle \rangle^2 \ (x \in \mathbb{R}^t).$$

**Lemma 2.3.** [1, Lemma 2.2] If $t > 2$ and $\mu \neq -1$ or 0, the functions $F_1, \ldots, F_n$ are linearly independent.

Since $\dim Q_t = \binom{t}{2} + t$, we deduce that $n \leq \binom{t}{2} + t$, equivalently $k \leq \binom{t}{2}$. The following result enables us to dispose of the regular graphs for which this bound is attained.

**Theorem 2.4.** [1, Theorem 3.1] Let $G$ be an $r$-regular graph $G$ of order $n$ with $\mu$ as an eigenvalue of multiplicity $k$. If $\mu \notin \{-1, 0, r\}$ and $t = n - k > 2$ then $k \leq \binom{t}{2} - 1$.

**Corollary 2.5.** If $G$ is a regular graph of order $\frac{1}{2}t(t + 1) \ (t > 2)$ with an eigenvalue $\mu \notin \{-1, 0\}$ of multiplicity $\frac{1}{2}t(t - 1)$ then $t = 3$, $\mu = 1$ and $G = 3K_2$.

**Proof.** If $G$ is $r$-regular then $\mu = r$ by Theorem 2.4, and so $G$ has $\frac{1}{2}t(t - 1)$ components, each with $\mu$ as a simple eigenvalue (cf. [4, Corollary 1.3.8]). It follows that $t \geq \frac{1}{2}t(t - 1)$, and hence that $t = 3$, $G = 3K_2$, $\mu = 1$. □
Next, using Equation (1), we see that
\[ \mu I - A = S^T (\mu I - C)^{-1} S, \]
and so, for all vertices \( u, v \) of \( G \),
\[ \langle s_u, s_v \rangle = \begin{cases} 
\mu & \text{if } u = v \\
-1 & \text{if } u \sim v \\
0 & \text{otherwise}
\end{cases} \quad (2) \]
It follows that if \( \mu \notin \{-1, 0\} \) then the \( H \)-neighbourhoods \( \Delta_H(u) \) \( (u \in X) \) are distinct and non-empty. When \( k = \binom{t}{3} \), our objective will be to show that, under suitable conditions, the \( H \)-neighbourhoods form a tight 4-design, that is, a design which satisfies the following conditions with \( s = 2 \).

**Theorem 2.6.** [2, Theorem 1.52] Let \( B \) be a collection of \( a \)-subsets of the \( t \)-set \( V \), where \( 2s \leq a \leq t - s \). Then any two of the following conditions imply the third.
(a) \((V, B)\) is a \( 2s \)-design;
(b) there are precisely \( s \) values for the numbers \(|B \cap B'|\), where \( B, B' \) are distinct sets in \( B \);
(c) \(|B| = \binom{t}{s}\).

Finally we can exploit the fact that tight 4-designs are extremely rare:

**Theorem 2.7.** [2, Theorem 1.54] Let \( D \) be a tight 4-\((t, a, l)\) design with \( 4 \leq a < t - 2 \). Then either \( D \) or its complement \( \overline{D} \) is the unique 4-\((23, 7, 1)\) design.

### 3 Proof of the main result

We retain the notation of the previous sections. Additionally we suppose that \( k = \frac{1}{2}t(t - 1) \) \( (t > 2) \), and that the star set \( X \) for \( \mu \neq -1, 0 \) is such that (i) \( X \cup \overline{X} \) is an equitable partition of \( G \), (ii) \( G_X \) is strongly regular with parameters \((n, r, e, f)\) \((0 < r < n - 1)\). We show first that \( t \neq 3 \) by inspecting the strongly regular graphs of order 6. If \( G_X = 2K_3 \) or \( \overline{2K_3} \) then there is no suitable bipartition \( X \cup \overline{X} \). If \( G_X = 3K_2 \) then \( G - X = K_1 \) and \( G = C_6 \), while if \( G = \overline{3K_2} \) then \( G - X = K_3 \) and \( G = \overline{C_6} \). In both cases \( G \) has no eigenvalue of multiplicity 3. Hence \( t > 3 \) and \( k > \frac{1}{2}n \). It follows that \( \mu \) is an integer, for otherwise \( \mu \) has an algebraic conjugate which is a second eigenvalue of multiplicity \( k \).

The partition \( X \cup \overline{X} \) determines divisors of \( G \) and \( G_X \), and we denote the corresponding divisor matrices by
\[ D = \begin{pmatrix} p & a \\ b & q \end{pmatrix}, \quad D^* = \begin{pmatrix} p & t - a \\ k - b & q \end{pmatrix}. \]
respectively. Note that $|\Delta_H(u)| = a$ for all $u \in X$, and that $1 < a < t - 1$.
In what follows, we write $\mathbf{j}$ for an all-1 vector (with length determined by context), and $A^*$ for the adjacency matrix of $G_X$. Additionally, $E$ denotes an eigenspace of $G$ and $E^*$ denotes an eigenspace of $G_X$.

**Lemma 3.1.** There exist integers $\lambda, \rho$ such that $G_X$ has spectrum $r, \lambda^{(t)}, \mu^{(k)}$ and $G$ has spectrum $\rho, \lambda^{(t-1)}, \mu^{(k)}$.

**Proof.** Let $\mathcal{V}$ be the subspace of $\mathbb{R}^n$ spanned by the characteristic vectors of $X$ and $X^*$, and let $\mathcal{W} = \mathcal{V}^\perp$. Note that for any eigenvalue $\nu$, $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{W} \cap \mathcal{E}^{*}(\nu)$ if and only if $\begin{pmatrix} x \\ -y \end{pmatrix} \in \mathcal{W} \cap \mathcal{E}^*(\nu)$. The graph $G$ has linearly independent eigenvectors $x_1, x_2 \in \mathcal{V}$ with corresponding eigenvalues those of $D^*$. Moreover, if $\rho$ is the largest eigenvalue of $G$ then we may take $Ax_1 = \rho x_1$, $A^* x_1 = r x_1, x_1 = \mathbf{j}$ (cf. [4, Theorem 3.9.9]). Since $\mathcal{E}(\mu)$ and $\mathcal{W}$ are subspaces of $x_1^*$, we have $\dim(\mathcal{E}(\mu) \cap \mathcal{W}) \geq k - 1$. On the other hand, $\dim \mathcal{E}^*(\mu) \leq k - 1$ by Lemma 2.3, and so we deduce that $\mathcal{E}^*(\mu) \subseteq \mathcal{W}$, $\dim(\mathcal{E}(\mu) \cap \mathcal{W}) = k - 1$, and $Ax_2 = \mu x_2$. Let $A^* x_2^* = \lambda x_2^*$. Then $\mu \neq \lambda = p + q - r \in \mathbb{Z}$ and $G_X$ has spectrum $r, \lambda^{(t)}, \mu^{(k)}$ (cf. [4, Section 3.6]). Note that $\lambda \neq r$ for otherwise $G_X = (t + 1)K_{r+1}$ and $\mu = -1$, contrary to assumption. We deduce that $G$ has spectrum $\rho, \lambda^{(t-1)}, \mu^{(k)}$. Finally, $\rho \in \mathbb{Z}$ because $p + q - \mu.$

When $G = G_{36}$ and $G_X = L(K_9)$, we have $\mu = -2$, $t = 8$, $k = 28$, $r = 14$, $\rho = 21$, $\lambda = 5$, $p = 12$, $q = 7$, $a = 6$, $b = 21$.

**Lemma 3.2.** The matrix $\mu^2 I + A$ is invertible.

**Proof.** Since $\mu^2 \notin \{-\rho, -\mu\}$, it suffices to show that $\mu^2 \neq -\lambda$. Now the multiplicities of $\lambda$ and $\mu$ in the strongly regular graph $G_X$ are given by

$$m(\lambda) = \frac{r(r - \mu)(\mu + 1)}{(r + \lambda \mu)(\mu - \lambda)}, \quad m(\mu) = \frac{r(r - \lambda)(\lambda + 1)}{(r + \lambda \mu)(\lambda - \mu)},$$

formulae which follow from [4, Theorems 3.6.4 and 3.6.5]. Suppose by way of contradiction that $\lambda = -\mu^2$. Then $\mu > 0$. Since $m(\lambda) = t$ and $m(\mu) = k - 1 = \frac{1}{2}(t + 1)(t - 2)$, we have:

$$\frac{(t + 1)(t - 2)}{2t} = \frac{m(\mu)}{m(-\mu)} = \frac{(r + \mu^2)(\mu - 1)}{r - \mu}.$$  \hspace{1cm} (3)

Let $\theta = (r - \mu)/t$. Then

$$0 = \text{tr}(A^*) = \mu + \theta t + t(-\mu^2) + \frac{1}{2}(t + 1)(t - 2)\mu,$$

whence $\theta = \mu^2 - \frac{1}{2}(t - 1)\mu$. Substituting $\mu + \mu^2 t - \frac{1}{2}t(1 - \mu)$ for $r$ in Equation (3), and dividing by $\mu(t + 1)$, we obtain:

$$(t - 2\mu)(t - 2\mu - 1) = 2(1 - \mu).$$  \hspace{1cm} (4)
Since \( \mu \in \mathcal{N} \), the left hand side of (4) is non-negative, while the right hand side is non-positive. We conclude that \( \mu = 1 \) and \( t \in \{2,3\} \), a contradiction.

\[ \square \]

We are now in a position to prove the following.

**Lemma 3.3.** If \( 4 \leq a \leq t - 2 \), the \( H \)-neighbourhoods \( \Delta_H(u) \) \( (u \in X) \) form a tight 4-design.

**Proof.** By Lemma 2.3, the functions \( \langle \langle s_u, x \rangle \rangle^2 \) \( (u \in X) \) form a basis for \( \mathcal{Q}_t \). Let

\[ \langle \langle x, x \rangle \rangle = \sum_{u=1}^{n} \gamma_u \langle \langle s_u, x \rangle \rangle^2, \quad (5) \]

and write \( c = (\gamma_1, \gamma_2, \ldots, \gamma_n)^T \). From Equation (2) we have

\[ \mu = \langle \langle s_i, s_i \rangle \rangle = \mu^2 \gamma_i + \sum_{u \sim i} \gamma_u \quad (i = 1, 2, \ldots, n), \]

whence \( (\mu^2 I + A) c = \mu j \). In view of Lemma 3.2, we have \( c = (\mu^2 I + A)^{-1} \mu j \). In the notation of Lemma 3.1, we have \( j \in \mathcal{V} \), while \( \mathcal{V} \) is \( A \)-invariant since the eigenvectors \( x_1^*, x_2^* \) form a basis for \( \mathcal{V} \). It follows that there exist \( \xi, \eta \in \mathbb{R} \) such that \( c = \left( \begin{array}{c} \xi_j \\ \eta_j \end{array} \right) \).

We extend notation in a natural way, so that for example \( \Delta_H^*(u) \) denotes the set of vertices in \( \overline{X} \) adjacent to \( u \) in \( G_{\overline{X}} \). For \( i, j \in X \), let \( r_{ij} = |\Delta_X(i) \cap \Delta_X(j)| \), \( s_{ij} = |\Delta_H(i) \cap \Delta_H(j)| \) and \( t_{ij} = |\Delta_H^*(i) \cap \Delta_H^*(j)| \). Note that \( r_{ij} = |\Delta_X(i) \cap \Delta_X^*(j)| \) and \( t_{ij} = t - 2a + s_{ij} \). Since \( r_{ij} + t_{ij} \) is the number of common neighbours of \( i \) and \( j \) in \( G_{\overline{X}} \), we have

\[ r_{ij} + s_{ij} = \begin{cases} 2a - t + e & \text{if } i \sim j \\ 2a - t + f & \text{if } i \not\sim j \end{cases}. \quad (6) \]

Since \( \langle \langle x, y \rangle \rangle = \frac{1}{5}(\langle \langle x + y, x + y \rangle \rangle - \langle \langle x - y, x - y \rangle \rangle) \), Equation (5) yields

\[ \langle \langle x, y \rangle \rangle = \sum_{u=1}^{n} \gamma_u \langle \langle s_u, x \rangle \rangle \langle \langle s_u, y \rangle \rangle. \]

Setting \( x = s_i \), \( y = s_j \) we obtain:

\[ -1 = \xi(r_{ij} - 2\mu) + \eta s_{ij} \quad \text{if } i \sim j, \quad 0 = \xi r_{ij} + \eta s_{ij} \quad \text{if } i \not\sim j. \quad (7) \]

Thus from (6) and (7) we obtain two sets of simultaneous equations in \( r_{ij} \) and \( s_{ij} \):

\[
\begin{align*}
 r_{ij} + s_{ij} &= 2a - t + e & \text{if } i \sim j, \\
 \xi r_{ij} + \eta s_{ij} &= 2\mu \xi - 1 & \text{if } i \sim j,
\end{align*}
\]

\[
\begin{align*}
 r_{ij} + s_{ij} &= 2a - t + f & \text{if } i \not\sim j, \\
 \xi r_{ij} + \eta s_{ij} &= 0 & \text{if } i \not\sim j.
\end{align*}
\]

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Now $\xi \neq \eta$ for otherwise $j$ is an eigenvector of $A$, and $G$ is regular, contradicting Corollary 2.5. Therefore each set of simultaneous equations has a unique solution; in particular, there exist integers $e', f'$ such that

$$|\Delta_H(i) \cap \Delta_H(j)| = \begin{cases} e' & \text{if } i \sim j, \\ f' & \text{if } i \not\sim j. \end{cases}$$

We have $e' \neq f'$, for otherwise $|X| \leq t$ [2, Theorem 1.51]. Thus if $4 \leq a \leq t - 2$ then by Theorem 2.7 the $H$-neighbourhoods $\Delta_H(u)$ ($u \in X$) form a tight 4-design.

In view of Theorem 2.7, it remains to consider four cases: (a) $t = 23$ and $a \in \{7, 16\}$, (b) $a = 3$, (c) $a = 2$, (d) $a = t - 2$.

Case (a). In this case we have $n = 276$, $|X| = 253$, $|\overline{X}| = 23$ and either $a = 7$ or $a = 16$. If $a = 7$ then $D^* = \begin{pmatrix} r - 16 & 16 \\ 176 & r - 176 \end{pmatrix}$, whence $176 \leq r \leq 198$. If $a = 16$ then $D^* = \begin{pmatrix} r - 7 & 7 \\ 77 & r - 77 \end{pmatrix}$, whence $77 \leq r \leq 99$. For these values of $r$, there is no strongly regular graph of order 276 and degree $r$; see for example Brouwer’s list of feasible parameters at http://www.win.tue.nl/~aeb/graphs/srg/srgtab.html.

Case (b): $a = 3$. Let $D = \{\Delta_H(u) : u \in X\}$. If $t \geq 7$ then $\overline{D}$ is a tight 4-design and Theorem 2.7 is contradicted. If $t < 7$ then $t \in \{5, 6\}$ and the multiplicity of $\mu$ in $G_X$ is 9, 14 respectively. If $t = 5$ then either $G_X = L(K_6)$ with $\mu = -2$ or $G_X = \overline{L}(K_6)$ with $\mu = 1$. In the former case, $D^* = \begin{pmatrix} 6 & 2 \\ 4 & 4 \end{pmatrix}$, and so $H = K_5$; but the graph obtained from $K_5$ by adding a vertex of degree 3 does not have $-2$ as an eigenvalue (see Lemma 2.2). In the latter case, $D^* = \begin{pmatrix} 4 & 2 \\ 4 & 2 \end{pmatrix}$, and so $H$ is a 5-cycle; but then not all 3-subsets of $\overline{X}$ can be $H$-neighbourhoods in $G$ [4, Example 5.2.3]. Now suppose that $t = 6$. Then $k = 15$ and we have $b = ka/t = 45/6$, a contradiction.

Case (c): $a = 2$. Here the $H$-neighbourhoods in $G$ are all the 2-subsets of $\overline{X}$, and their intersection numbers are necessarily 0 and 1. We have

$$D^* = \begin{pmatrix} r - t + 2 & t - 2 \\ \frac{1}{2}(t - 1)(t - 2) & r - \frac{1}{2}(t - 1)(t - 2) \end{pmatrix},$$

whence $\lambda = r - \frac{1}{2}(t - 2)(t + 1)$. Now

$$0 = \text{tr}(A^*) = r + t(r - \frac{1}{2}(t - 2)(t + 1)) + \frac{1}{2}(t + 1)(t - 2)\mu,$$

whence

$$\mu = t - \frac{2r}{t - 2}. \quad (8)$$
A 2-subset of $X$ intersects precisely $2t - 4$ other 2-subsets of $X$. Hence if $(e', f') = (1, 0)$ then each vertex in $X$ has $2t - 4$ neighbours in $X$, and so $2t - 4 = r - t + 2$. Thus $r = 3(t - 2)$, and we see from Equation (8) that $\mu = t - 6$. Now $r \geq \frac{1}{2}(t - 1)(t - 2)$ and so $t \leq 7$. Since $\mu \neq -1$ or 0, we have $t = 4$ or $t = 7$. If $t = 4$ then

$$D^* = \begin{pmatrix}
4 & 2 \\
3 & 3 
\end{pmatrix} = D,$$

whence $G$ is 6-regular, a contradiction. If $t = 7$ then $r = 15$, $\mu = 1$ and $X$ is an independent set. Equating diagonal entries in Equation (1), we find that $a = 1$, a contradiction.

If $(e', f') = (0, 1)$ then each vertex in $X$ has $2t - 4$ non-neighbours in $X$, and so $2t - 4 = \frac{1}{2}t(t - 1) - 1 - (r - t + 2)$. We find that $r = \frac{1}{2}(t - 1)(t - 2)$, $\mu = 1$ and $X$ is independent, leading to the same contradiction as above.

**Case (d):** $a = t - 2$. In this case we have $D^* = \begin{pmatrix}
r - 2 & 2 \\
t + 1 & r - t + 1 
\end{pmatrix}$, whence $\lambda = r - t - 1$. Now

$$0 = tr(A^*) = r + t(r - t - 1) + \frac{1}{2}(t + 1)(t - 2)\mu,$$

and so

$$\mu = \frac{2(t - r)}{t - 2}.$$

For distinct $u, v \in X$, let

$$|\Delta^*_H(u) \cap \Delta^*_H(v)| = \begin{cases}
eu* \text{ if } u \sim v, \\
f^* \text{ if } u \not\sim v,
\end{cases}$$

so that \{e^*, f^*\} = \{0, 1\}.

If $(e^*, f^*) = (0, 1)$ then each vertex of $X$ has $2t - 4$ non-neighbours in $X$, and so $r - 2 = \left(\frac{t}{2}\right) - (2t - 4) - 1$, whence $r = \frac{1}{2}(t^2 - 5t + 10)$ and $\mu = 5 - t$. Since $r - t + 1 \leq t - 1$, we have $(t - 2)(t - 7) \leq 0$, whence $t \in \{4, 5, 6, 7\}$. If $t = 4$ then $r = 3$, $\mu = 1$ and $G$ is 3-regular, contradicting Theorem 2.4. If $t = 5$ or 6 then $\mu = 0$ or $-1$, contrary to assumption. If $t = 7$ then $a = 5$, $\mu = -2$, $H = K_7$ and we obtain a contradiction from Lemma 2.2.

If $(e^*, f^*) = (1, 0)$ then each vertex of $X$ has $2t - 4$ neighbours in $X$, and so $r = 2t - 2$, $\mu = -2$. Moreover, $H = K_t$ and Lemma 2.2 yields

$$(t - 2)^2 - (t + 1)(t - 2) + 2(t + 1) = 0.$$ 

It follows that $t = 8$. Since $G$ is determined by $H$ and all 6-subsets of $X$ as $H$-neighbourhoods of vertices in $X$, we conclude that $G = G_{36}$.

This completes the proof of Theorem 1.1

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References


