Star complements and exceptional graphs

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Abstract

Let \( G \) be a finite graph of order \( n \) with an eigenvalue \( \mu \) of multiplicity \( k \). (Thus the \( \mu \)-eigenspace of a \((0, 1)\)-adjacency matrix of \( G \) has dimension \( k \).) A star complement for \( \mu \) in \( G \) is an induced subgraph \( G - X \) of \( G \) such that \( |X| = k \) and \( G - X \) does not have \( \mu \) as an eigenvalue. An exceptional graph is a connected graph, other than a generalized line graph, whose eigenvalues lie in \([-2, \infty)\). We establish some properties of star complements, and of eigenvectors, of exceptional graphs with least eigenvalue \(-2\).

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1. Introduction

Let \( G \) be a finite graph of order \( n \) with an eigenvalue \( \mu \) of multiplicity \( k \). (Thus the corresponding eigenspace of a \((0, 1)\)-adjacency matrix of \( G \) has dimension \( k \).) A star set for \( \mu \) in \( G \) is a set \( X \) of \( k \) vertices in \( G \) such that the induced subgraph \( G - X \) does not have \( \mu \) as an eigenvalue. In this situation, \( G - X \) is called a star complement for \( \mu \) in \( G \) (or in [15] a \( \mu \)-basic subgraph of...
Star sets and star complements exist for any eigenvalue of any graph, and serve to explain the relation between graph structure and a single eigenvalue \( \mu \). When \( \mu \) is not \(-1\) or \(0\), they can be used to determine sharp upper bounds for \( k \) in arbitrary graphs and in regular graphs [2]; to characterize certain graphs (see for example [12]); and to find all the exceptional graphs (i.e. connected graphs with all eigenvalues at least \(-2\) that are not generalized line graphs) [9]. There are also connections with dominating properties [10, Section 7.6] and independent sets [19]. For a recent survey, see [18] and for basic properties, see [13, Chapter 5]. Here we investigate properties of star sets and star complements related to graphs with least eigenvalue \(-2\), and explain some phenomena observed from earlier computer results. Explicitly, we give a simple computer-free proof that each exceptional graph with least eigenvalue greater than \(-2\) is an induced subgraph of an exceptional graph with least eigenvalue equal to \(-2\); we show how extendability graphs [13, Section 5.1] can be used to investigate the regular exceptional graphs; and we establish a property of eigenvectors of exceptional graphs with \(-2\) as a simple eigenvalue.

The following result [13, Theorem 5.1.7] establishes the fundamental property of star complements: if \( X \) is a star set for \( \mu \) in \( G \), and if \( H \) is the star complement \( G - X \), then \( G \) is determined by \( \mu \), \( H \) and the \( H \)-neighbourhoods of vertices in \( X \).

**Theorem 1.1.** Let \( X \) be a set of \( k \) vertices in the graph \( G \) and suppose that \( G \) has adjacency matrix \( \begin{pmatrix} A_X & B \\ B^T & C \end{pmatrix} \), where \( A_X \) is the adjacency matrix of the subgraph induced by \( X \). Then \( X \) is a star set for \( \mu \) in \( G \) if and only if \( \mu \) is not an eigenvalue of \( C \) and

\[
\mu I - A_X = B^T (\mu I - C)^{-1} B.
\]

In this situation, the eigenspace of \( \mu \) consists of the vectors \( \begin{pmatrix} x \\ (\mu I - C)^{-1} Bx \end{pmatrix} \), where \( x \in \mathbb{R}^k \).

Recall that \( \mu \) is a main eigenvalue of \( G \) if the eigenspace \( \mathcal{E}(\mu) \) is not orthogonal to the all-1 vector \( j \).

In Section 2, we discuss the addition of a vertex to an exceptional star complement for \(-2\) to obtain \(-2\) as a main eigenvalue, and the addition of a star set to obtain \(-2\) as a non-main eigenvalue. In Section 3 we discuss integral eigenvectors of exceptional graphs having \(-2\) as a simple eigenvalue.

### 2. Eigenvalues of exceptional graphs

We denote the least eigenvalue of a graph \( G \) by \( \lambda(G) \). Let \( \mathcal{H} \) denote the family of 443 exceptional graphs of order 8 with \( \lambda(G) > -2 \). These graphs were found by Doob and Cvetković [14] in 1979, and are listed as \( H001, \ldots, H443 \) in [13, Table A2]. The 473 maximal exceptional graphs were found by computer in 1999, as described in [9], and from these calculations we know that each graph in \( \mathcal{H} \) arises as a star complement for \(-2\). We begin by verifying this observation theoretically; the desirability of a computer-free proof was noted in [1, p. 17]. Since any exceptional graph \( G \) with \( \lambda(G) > -2 \) is an induced subgraph of a graph in \( \mathcal{H} \), one consequence of the result is that no exceptional graph \( G \) with \( \lambda(G) > -2 \) is a maximal exceptional graph.

**Proposition 2.1.** If \( H \in \mathcal{H} \) then \( H \) has a one-vertex extension \( H' \) with \( \lambda(H') = -2 \).

**Proof.** By [13, Theorem 2.3.19], \( H \) has an exceptional induced subgraph of order 6, and so there exists a vertex \( u \) of \( H \) such that \( H - u \) is exceptional. In the terminology of [13, Section 3.7], \( H \) generates the root system \( E_8 \), while \( H - u \) generates \( E_7 \). If the graph \( G \) generates \( E_k (k \in \{6, 7, 8\}) \)
then its adjacency matrix has the form \( Q^T Q - 2I \), where \( Q^T Q \) is the Gram matrix of an integral basis for the integral lattice \( L(E_k) \) generated by \( E_k \). Now the determinant of such a Gram matrix is a lattice invariant called the \textit{discriminant} of \( L(E_k) \), shown in [3, Section 3.10] to be 1 when \( k = 8, 2 \) when \( k = 7 \), and 3 when \( k = 6 \). Thus \( P_G(-2) = \det(-Q^T Q) = (-1)^k(9 - k) \) (cf. [7, Theorem 3], [13, Lemma 7.5.2]).

Now let \( H' \) be the graph obtained from \( H \) by attaching a pendant vertex at \( u \). From [6, Theorem 2.11], the characteristic polynomial of \( H' \) is given by

\[
P_{H'}(x) = x P_H(x) - P_{H-u}(x).
\]

In view of the preceding remarks, we have \( P_H(-2) = 1 \) and \( P_{H-u}(-2) = -2 \), and so from (2) we obtain \( P_{H'}(-2) = 0 \). By the Interlacing Theorem [13, Theorem 1.2.21], \( \lambda(H') = -2 \). □

In the foregoing proof, the appeal to the theory of lattices can be avoided by arguing as follows. If \( A \) is the adjacency matrix of \( H \) then \( A + 2I = Q^T Q \), where each column of the invertible \( 8 \times 8 \) matrix \( Q \) lies in \( E_8 \). The seven columns of \( Q \) corresponding to \( H - u \) lie in a subsystem \( E_7 \) which consists of the vectors in \( E_8 \) orthogonal to a fixed vector \( b \) of \( E_8 \). Since \( E_8^\perp = \{0\} \), \( b \) is not orthogonal to the remaining column \( q \) of \( Q \). Replacing \( b \) with \(-b\) if necessary, we may assume that \( b \cdot q = 1 \). Now let \( R = [Q|b] \). Then \( R^T R = A' + 2I \), where \( A' \) is the adjacency matrix of \( H' \). Since \( R^T R \) has rank 8, \(-2\) is an eigenvalue of \( H' \), and by the Interlacing Theorem, \( \lambda(H') = -2 \) as required.

Given a representation of \( H' \) in \( E_8 \), it can be shown by the methods of [11, Section 2] that \( H' \) has a one-vertex extension for which \(-2\) is an eigenvalue of multiplicity 2.

The graph \( H' \) has \( H \) as a star complement for \(-2\), but the eigenvalue \(-2\) of \( H' \) may or may not be a main eigenvalue. If \( v \) is an eigenvector of \( H' \) corresponding to \(-2\) then \( Rv = 0 \), while \(-2\) is a main eigenvalue if and only if \( v \cdot j_9 \neq 0 \). Let \( R' \) be the matrix obtained from \( R \) by adding \( j_9^* \) as a ninth row. Then \(-2\) is a non-main eigenvalue if and only if \( R'v = 0 \), equivalently \( j_9 \) lies in the column space of \( R^T \).

For an investigation of main and non-main eigenvalues, we use the notation of Section 1 with \( t = n - k \) and \( j = j_1. \) Let \( \{e_1, \ldots, e_k\} \) be the standard orthonormal basis of \( \mathbb{R}^k \), and define a bilinear form on \( \mathbb{R}^t \) by

\[
\langle x, y \rangle = x^T (\mu I - C)^{-1} y \quad (x, y \in \mathbb{R}^t).
\]

By Theorem 1.1, \( \mu \) is a non-main eigenvalue of \( G \) if and only if \( j_n \) is orthogonal to the vectors \( (\mu I - C)^{-1} e_i \) \((i = 1, \ldots, k) \). Since \( Be_i \) is the \( i \)-th column of \( B \), \( \mu \) is a non-main eigenvalue of \( G \) if and only if \( \langle b, j \rangle = -1 \) for each column \( b \) of \( B \). The computer calculations described in [9] show that each graph \( H \) in \( \mathcal{H} \) arises as a star complement for \(-2\) in a graph for which \(-2\) is a main eigenvalue [5, Theorem 11]. It follows from the foregoing remarks that each such graph \( H \) has a one-vertex extension for which \(-2\) is a main eigenvalue; in other words, there exists a column \( b \) such that \( \langle b, b \rangle = -2 \) and \( \langle b, j \rangle \neq -1 \). This fact has not yet been established theoretically.

The \textit{extendability graph} \( \Gamma(H; \mu) \) [13, p. 121] has as vertices the \((0,1)\)-vectors \( b \in \mathbb{R}^t \) such that \( \langle b, b \rangle = \mu \), with an edge between \( b \) and \( b' \) if and only if \( \langle b, b' \rangle \in (-1, 0] \). A clique on \( b_1, b_2, \ldots, b_k \) in \( \Gamma(H; \mu) \) determines a graph \( G \) with \( H \) as a star complement for \( \mu \); in the notation of Theorem 1.1, \( H = G - X \) where \( X = \{1, 2, \ldots, k\} \), \( B = [b_1 | b_2 | \cdots | b_k] \), and vertices \( i, j \) of \( X \) are adjacent if and only if \( \langle b_i, j \rangle = -1 \). We may define the \textit{non-main extendability graph} \( \Gamma^*(H; \mu) \) as the subgraph of \( \Gamma(H; \mu) \) induced by those \((0,1)\)-vectors \( b \) for which \( \langle b, j \rangle = -1 \).
Proposition 2.2. Let $H \in \mathcal{H}$ and suppose that the cone $K_1 \nabla H$ has $-2$ as an eigenvalue. Then $\Gamma^*(H; -2)$ has a perfect matching, say $b_1c_1, \ldots, b_mc_m$, with $b_i + c_i = j$ ($i = 1, \ldots, m$). Moreover the following hold.

(i) If $\Gamma^*(H; -2)$ has a clique of order $m$ then $\Gamma^*(H; -2)$ is a cocktail-party graph $CP(2m) = mK_2$. In this situation every maximal clique has order $m$, there are $2^m$ maximal cliques, and the $2^m$ corresponding graphs with $H$ as a star complement for $-2$ are switching-equivalent.

(ii) $m \leq 20$ and if $G$ has $H$ as a star complement for $-2$ as a non-main eigenvalue then $G$ is switching-equivalent to an induced subgraph of $L(K_8)$.

Proof. Since $K_1 \nabla H$ has $-2$ as an eigenvalue, we have $(j, j) = -2$. Hence, for any vertex $b$ of $\Gamma^*(H; -2)$ we have $(j - b, j - b) = -2$, so that $j - b$ is a vertex of $\Gamma(H; -2)$. In addition we have $(j - b, j) = -1$, and so $j - b$ is a vertex of $\Gamma^*(H; -2)$. Since $(j - b, b) = 1$, it follows that $b$ and $j - b$ are non-adjacent vertices of $\Gamma^*(H; -2)$, and hence that $\Gamma^*(H; -2)$ has a perfect matching with the property claimed.

For (i), note first that a clique of order $m$ in $\Gamma^*(H; -2)$ has precisely one vertex from each pair $(b_i, c_i)$; without loss of generality, $b_1, \ldots, b_m$ induce a clique $K$. Secondly, note that the map $b \leftrightarrow j - b$ is an isomorphism from the graph on $b_1, \ldots, b_m$ to the graph on $c_1, \ldots, c_m$, and so $c_1, \ldots, c_m$ also induce a clique. Thirdly, if $\langle b_i, b_j \rangle = 0$ then $\langle b_i, c_j \rangle = -1$, while if $\langle b_i, b_j \rangle = -1$ then $\langle b_i, c_j \rangle = 0$. Thus $\Gamma^*(H; -2) \cong CP(2m)$. Hence there are $2^m$ cliques of order $m$, each obtained from $K$ by replacing $b_i$ with $c_i$ for all $i$ in some subset of $\{1, \ldots, m\}$. As noted in [13, Section 5.5], the corresponding graphs with $H$ as a star complement for $-2$ are switching-equivalent.

For (ii), note that we may add to $G$ a vertex $v$ adjacent to every vertex of $G$ to obtain a cone $K_1 \nabla G$ which has $H$ as a star complement for $-2$. Now we argue as in [13, Proposition 6.2.1]: by choosing a suitable representation of $K_1 \nabla G$ in the root system $E_8$, we see that $G$ is switching-equivalent to an induced subgraph of $L(K_8)$. In particular, $v$ can have at most 28 neighbours and so $m \leq 20$. □

Example 2.3. Let $H$ consist of disjoint cycles of lengths 3 and 5 together with a bridge between them, i.e. $H$ is the exceptional graph $H010$. It is straightforward to verify that $K_1 \nabla H$ has $-2$ as an eigenvalue. Moreover, $H$ is an induced subgraph of a Chang graph $G$; and since $G$ is regular of order 28, $\Gamma^*(H; -2)$ has a clique of order 20 (necessarily maximal by Proposition 2.2(ii)). By Proposition 2.2(ii), $\Gamma^*(H; -2) \cong CP(40)$ and any maximal graph having $H$ as a star complement for $-2$ as a non-main eigenvalue is switching-equivalent to $G$, hence to $L(K_8)$.

The arguments of Example 2.3 apply whenever (i) $(j, j) = -2$, and (ii) $H$ is a star complement for $-2$ in a graph of order 28 with $-2$ as a non-main eigenvalue. M. Lepović (private communication) has verified by computer that exactly 198 of the 443 graphs $H \in \mathcal{H}$ satisfy condition (i), i.e. are such that the cone $K_1 \nabla H$ has $-2$ as an eigenvalue. The computer investigations reported in [8] show that all but one ($H434$) are star complements for $-2$ in some $K_1 \nabla G$, where $G$ is a Chang graph. Moreover 172 of the remaining 197 graphs have maximal degree less than 7, hence are induced subgraphs of a Chang graph $G$. Thus for each of these 172 graphs $H$ we have $\Gamma^*(H; -2) \cong CP(40)$. The same holds when $H$ is the graph $H440$ (with maximal degree 7): in this case, there are many non-isomorphic graphs among the corresponding $2^{20}$ graphs of order 28 that have $-2$ as a non-main eigenvalue. (In [5, Section 6] it was asserted wrongly that only one such graph of order 28 exists; however Theorem 11 of [5] remains valid.)
In the next example, we construct \( \Gamma^*(H; -2) \) explicitly when \( H \) is \( H443 \), another graph in \( \mathcal{H} \) with maximal degree 7. In this case we do not have prior knowledge of a graph of order 28 with \(-2\) as a non-main eigenvalue. The calculations show that there is no regular graph with \( H443 \) as a star complement for \(-2\).

**Example 2.4.** Let \( H \) be the complement of \( K_{1,2} \cup 5K_1 \), i.e. \( H \) is the exceptional graph \( H443 \) which features as a versatile star complement for \(-2\) in [13, Section 6.3]. A vertex \( u \) in \( X \) is said to be of type \( abc \) if its \( H \)-neighbourhood \( \Delta_H(u) \) consists of \( a, b, c \) vertices of degree 5, 6, 7 respectively. Let \( C \) be the adjacency matrix of \( H \), with vertices labelled so that their degrees are in non-decreasing order. To use Theorem 1.1 with \( \mu = -2 \), note that

\[
(2I_8 + C)^{-1} = \begin{pmatrix} 8 & 5J_2^T & -3J_3^T \\ 5J_2 & 3J_{2,2} + I_2 & 2J_{2,5} \\ -3J_3 & -2J_{3,2} & J_{5,5} + I_5 \end{pmatrix},
\]

where \( I_m \) denotes the \( m \times m \) identity matrix and \( J_{m,n} \) denotes the all-1 matrix of size \( m \times n \). If we now equate diagonal entries in Eq. (1), we obtain

\[
2 = a + b + c + 7a^2 + 10ab - 6ac - 4bc + 3b^2 + c^2. \tag{3}
\]

If \( u, v \) are distinct vertices of types \( a_1b_1c_1, a_2b_2c_2 \) then, equating off-diagonal entries in Eq. (1), we have

\[
a_{uv} = |\Delta_H(u) \cap \Delta_H(v)| + 7a_1a_2 + 3b_1b_2 + c_1c_2 + 5(a_1b_2 + a_2b_1) - 3(a_1c_2 + a_2c_1) - 2(b_1c_2 + b_2c_1). \tag{4}
\]

The 10 solutions of Eq. (3) are given in [13, Table 1, p. 148] along with information from Eq. (4) sufficient to construct \( \Gamma(H; -2) \). Since \( (a, b, c) = (1, 2, 5) \) is one solution of (3), \( j \) arises as a vertex and its neighbours include the vertices of \( \Gamma^*(H; -2) \). The neighbours \( b \) of \( j \) for which \( (b, j) = -1 \) can be identified from Eq. (4). They correspond to ten vertices in \( X \) of type 011, ten of type 114, ten of type 023 and ten of type 102. (Note that if \( b \) is of type 011 or 023 then \( j - b \) is of type 114 or 102, respectively.) We deduce that again \( \Gamma^*(H; -2) \cong CP(40) \), and so we obtain 2\(^{20} \) maximal graphs with \( H \) as a star complement for \(-2\) as a non-main eigenvalue. M. Lepočić (private communication) has shown by computer that 356 non-isomorphic graphs arise in this way. We note in passing that the cones over \( H \) and any of these graphs of order 28 not only have \(-2\) as a main eigenvalue but also have \( K_8 \) (a subgraph of \( K_1 \) as a star complement for \(-2\)).

None of the graphs of order 28 here is regular; indeed we can show as follows that there is no regular graph \( G \) with \( H \) as a star complement for \(-2\). For suppose that \( H = G - X \) where \( G \) is \( r \)-regular and the star set \( X \) consists of \( e, f, g, h \) vertices of type 102, 114, 023, 011 respectively. Let \( e + g = p, f + h = q \) and note that \( 0 \leq p \leq 10, 0 \leq q \leq 10 \). Counting edges between \( X \) and vertices in \( H \) of degree 5, 6, 7 in turn, we have:

\[
r = 5 + e + f, \quad 2r = 12 + f + 2g + h, \quad 5r = 35 + 2e + 4f + 3g + h.
\]

We can now write \( e = p - g, h = q - f \) and solve these equations for \( r, f, g \) in terms of \( p \) and \( q \). We find that \( f = \frac{1}{2}(q - 14) \), a contradiction.

We know that every exceptional graph \( G \) with least eigenvalue \(-2\) has an exceptional star complement \( H \) for \(-2 \) [13, Theorem 5.31]. We shall see that if \( G \) is regular and \( H \) has order 8 then \( H \) satisfies the hypotheses of Proposition 2.2.
Lemma 2.5. Let $G$ be an $r$-regular graph of order $n$, and let $\mu$ be an eigenvalue of $G$ other than $r$. If $C$ is an adjacency matrix of any star complement for $\mu$, then

$$j^T(\mu I - C)^{-1}j = \frac{n}{\mu - r}.$$ 

Proof. In the notation of Theorem 1.1, let $S = (B|C - \mu I)$. Then Eq. (1) may be written

$$\mu I - A = S^T(\mu I - C)^{-1}S.$$ 

Now $S_jn = (r - \mu)j$ and so $j^T_n(\mu I - A)j_n = (r - \mu)^2j^T(\mu I - C)^{-1}j$. The result follows since $j^T_n(\mu I - A)j_n = \mu n - rn$. \qed

Now we apply Lemma 2.5 in the case that $G$ is an exceptional regular graph, $\mu = -2$, and $C$ is the adjacency matrix of an exceptional star complement $H$ for $-2$. Then $H$ has order $k \in \{6, 7, 8\}$ and $A + 2I = M^TM$ for some $k \times n$ matrix $M$ of rank $k$: the columns of $M$ are a representation of $G$ in the root system $E_t$ (cf. [13, Chapter 3]). Let $u = \frac{1}{\nu_{n+2}}Mj$, so that $u^Tu = \frac{n}{\nu_{n+2}}$. From the proof of [13, Theorem 4.1.5] we know that $u^Tu \in \{2, \frac{4}{3}\}$ if $t = 6$, $u^Tu \in \{2, \frac{3}{2}\}$ if $t = 7$ and $u^Tu = 2$ if $t = 8$. Thus necessarily $n = 2(r + 2)$ when $k = 8$; in this case we have $(j, j) = -2$ by Lemma 2.5, and so $H$ satisfies the hypotheses of Proposition 2.2. Accordingly, we have the following result.

Theorem 2.6. Let $G$ be an exceptional regular graph with least eigenvalue $-2$, having $H$ as an exceptional star complement for $-2$. If $H$ has order 8, then

(i) $H$ is one of 198 graphs in $\mathcal{H}$,

(ii) both $K_1 \nabla H$ and $K_1 \nabla G$ have $-2$ as a main eigenvalue,

(iii) $G$ is switching-equivalent to an induced subgraph of $L(K_8)$.

Example 2.7. Consider a cone $K_1 \nabla G$ such that some star complement $H$ for $-2$ in $G$ is also a star complement for $-2$ in $K_1 \nabla G$; for example, 430 of the 432 maximal exceptional graphs of order 29 have this property (see [13, Section 6.1]. By [10, Eq. (4.3.7)], $-2$ is a non-main eigenvalue of $G$, a fact we can also establish as follows. The extendability graph $\Gamma(H; -2)$ has $j$ as a vertex such that $(b, j) = -1$ for all other vertices $b$; thus deletion of the vertex of the cone leaves $G$ as a graph in which $-2$ is a non-main eigenvalue. Moreover (cf. [13, Section 5.5]) all vertices of $G$ outside $H$ are amenable to switching, and any switching yields another graph $G'$ with $-2$ as a non-main eigenvalue. If $K_1 \nabla G'$ is a maximal exceptional graph, then $G'$ is a maximal graph with $H$ as a star complement for $-2$ as a non-main eigenvalue.

Further remarks and examples may be found in [5, Section 6].

3. Eigenvectors of exceptional graphs

In this section we discuss exceptional graphs with $-2$ as a simple eigenvalue. Such graphs have a star complement for $-2$ of order 6, 7 or 8, and the eigenvectors corresponding to $-2$ are all scalar multiples of an eigenvector $v$ whose entries are integers. If $v$ is chosen with minimal norm, then $v$ is called a minimal integral eigenvector, and its height is the maximum modulus of its coordinates. We establish theoretically a property of heights noted from computer results given in [5].
First we give a short proof of the following theorem, established in [17] in a chemical context, and generalized in [16].

**Theorem 3.1.** Let $\lambda$ be a simple eigenvalue of the graph $G$. Then there exists an eigenvector $x = (x_1, \ldots, x_n)^T$ corresponding to $\lambda$ such that

$$x_j^2 = |P_{G-j}(\lambda)| \quad (j = 1, \ldots, n).$$

**Proof.** Let $\mu_1, \ldots, \mu_m$ be the distinct eigenvalues of $G$, so that [10, Section 4.2]

$$P_{G-j}(x) = P_G(x) \sum_{i=1}^m \frac{\|P_i e_j\|^2}{x - \mu_i} \quad (j = 1, \ldots, n),$$

where $P_i$ is the orthogonal projection of $\mathbb{R}^n$ onto $\mathcal{E}(\mu_i)$ and $\{e_1, \ldots, e_n\}$ is the standard orthonormal basis of $\mathbb{R}^n$. Now suppose that $\lambda = \mu_h$, so that

$$\|P_h e_j\|^2 = \frac{P_{G-j}(\lambda)}{P_G(\lambda)} \quad (j = 1, \ldots, n).$$

On the other hand, $\|P_h e_j\|^2 = e_j^T P_h e_j$, the $(j, j)$-entry of $P_h$; and if $u = (u_1, \ldots, u_n)^T$ is a unit eigenvector which spans $\mathcal{E}(\lambda)$ then $P_h = uu^T$, with $(j, j)$-entry $u_j^2$. The result follows by defining

$$x_j = \sqrt{|P_{G-j}(\lambda)| u_j} \quad (j = 1, \ldots, n). \quad \square$$

Following [7], we define the discriminant $d_G$ of a graph $G$ with $\lambda(G) \geq -2$ as $(-1)^n P_G(-2)$; and for $k = 6, 7, 8$, we define $\mathcal{G}_k$ as the set of exceptional graphs on $k$ vertices with $\lambda(G) > -2$. (Thus $\mathcal{G}_8 = \emptyset$.) As we saw in the proof of Proposition 2.1, if $G$ belongs to $\mathcal{G}_k$ then $d_G = 9 - k$.

Let $\mathcal{G}_k^*$ be the set of graphs which have $-2$ as a simple eigenvalue and a graph in $\mathcal{G}_k$ as a star complement for $-2$. (It is noted in [5] that $|\mathcal{G}_8^*| = 51$, $|\mathcal{G}_7^*| = 512$ and $|\mathcal{G}_6^*| = 4206$.)

**Corollary 3.2.** If $G$ belongs to $\mathcal{G}_k^*$, then $G$ has an integral minimal eigenvector corresponding to $-2$ with a coordinate equal to $1$.

**Proof.** By Theorem 3.1, the simple eigenvalue $-2$ of $G$ has an eigenvector $x$ such that $|x_i| = \sqrt{d_{G-i}}(i = 1, \ldots, n)$. For some $i$ the subgraph $G - i$ is an exceptional star complement for $-2$, so that (replacing $x$ with $-x$ if necessary) we have $x_i = \sqrt{9 - k}$. Hence $x = \sqrt{9 - k}x'$, where $x' = (x'_1, \ldots, x'_{k+1})^T$, and each $x'_j$ is rational. Now $\sqrt{9 - k} \in \{1, \sqrt{2}, \sqrt{3}\}$, while each $x'_j$ is an integer, and so each $x'_j$ is an integer. Since also $x'_1 = 1$, $x'$ is an eigenvector satisfying the conclusions of the Corollary. \(\square\)

Corollary 3.2 confirms an empirical observation from computer calculations of eigenvectors reported in [4]. We saw in the proof that $x_j = \sqrt{9 - k}x'_j$ where $x'_j$ is an integer $(j = 1, \ldots, k + 1)$, and so we can also deduce the following result from Theorem 3.1.

**Corollary 3.3.** If $G$ belongs to $\mathcal{G}_k^*$, then each $|P_{G-j}(-2)| (j = 1, \ldots, k + 1)$ is of the form $(9 - k)s^2$ where $s$ is an integer.

**Proposition 3.4.** If $G$ belongs to $\mathcal{G}_k^*$ then the height of an integral minimal eigenvector is less than or equal to 3, 4, 6 for $k = 6, 7, 8$ respectively.
Proof. If \( G \) has an induced subgraph \( K \) isomorphic to \( K_{1,4} \) then for each vertex \( j \) outside \( K \), \( G - j \) has \(-2\) as an eigenvalue (by interlacing) and so the \( j \)th entry of an integral minimal eigenvector \( x = (x_1, \ldots, x_{k+1})^T \) is zero. Since \( 1 \) and \(-2\), or \(-1 \) and \( 2 \), are the components of an integral minimal eigenvector of \( K_{1,4} \), it follows that \( x \) has coordinates \( 0, 1, -2 \) or \( 0, -1, 2 \). Accordingly suppose that \( G \) has no induced \( K_{1,4} \). Then each \( G - i \) has at most three components. When \( x_i \neq 0 \) we find an upper bound for \( d_{G - i} \) as the product of the discriminants of the possible components. The possible values of the discriminant of a component of order \( t \) here are: \( 9 - t \) (\( t = 6, 7, 8 \)) for an exceptional graph, \( 4 \) for the line graphs of odd unicyclic graphs or line graphs of trees with one petal, and \( t + 1 \) for the line graphs of trees (cf. [7, Theorem 3]). By considering all distributions of the vertices of \( G - i \) among at most three components we find easily that \( d_{G - i} \) is at most \( 27, 36 \) or \( 48 \) depending on whether the order of \( G \) is \( 7, 8 \) or \( 9 \). (For example, 48 is the product of discriminants of line graphs of orders 2, 3 and 3.) In the notation of Corollary 3.3, we have \( 3s^2 \leq 27 \) when \( k = 6 \), \( 2s^2 \leq 36 \) when \( k = 7 \), and \( s^2 \leq 48 \) when \( k = 8 \). Thus the height of \( x \) is at most 3, 4 or 6 respectively. □

The bounds in Proposition 3.4 are attained by the exceptional Smith graphs (cf. [13, Section 3.4]). See [5, Table 1] for additional data (obtained by computer) on the heights of eigenvectors.

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References

