
This is the peer reviewed version of this article

NOTICE: this is the author’s version of a work that was accepted for publication in Linear Algebra and its Applications resulting from the publishing process, such as peer review, editing, corrections, structural formatting, and other quality control mechanisms may not be reflected in this document. Changes may have been made to this work since it was submitted for publication. A definitive version was subsequently published in Linear Algebra and its Applications, [VOL 444 (2014)] DOI: http://dx.doi.org/10.1016/j.laa.2013.11.036
Abstract

Let $G$ be a connected cubic graph of order $n$ with $\mu$ as an eigenvalue of multiplicity $k$. We show that (i) if $\mu \notin \{-1, 0\}$ then $k \leq \frac{1}{2}n$, with equality if and only if $\mu = 1$ and $G$ is the Petersen graph; (ii) If $\mu = -1$ then $k \leq \frac{1}{2}n + 1$, with equality if and only if $G = K_4$; (iii) If $\mu = 0$ then $k \leq \frac{1}{2}n + 1$, with equality if and only if $G = \overline{2K_3}$.

AMS Classification: 05C50

Keywords: cubic graph, eigenvalue, star complement.
1 Introduction

Let $G$ be a regular graph of order $n$ with $\mu$ as an eigenvalue of multiplicity $k$, and let $t = n - k$. Thus the corresponding eigenspace $E(\mu)$ of a $(0,1)$-adjacency matrix $A$ of $G$ has dimension $k$ and codimension $t$. From [1, Theorem 3.1], we know that if $\mu \not\in \{-1,0\}$ and $t > 2$ then $k \leq n - \frac{1}{2}(-1 + \sqrt{8n + 9})$, equivalently $k \leq \frac{1}{2}(t + 1)(t - 2)$. For cubic graphs, this quadratic bound improves an earlier cubic bound noted in [4, p.162]. In fact, when $\mu \neq 0$ and $G$ is connected, a linear bound follows easily from the equation $\text{tr}(A) = 0$. To see this, note first that if $k \geq \frac{1}{2}n$ then $\mu$ is an integer, for otherwise it has an algebraic conjugate which is a second eigenvalue of multiplicity $\frac{1}{2}n$. It follows that if $G$ is a connected cubic graph then $\mu \in \{-2,-1,0,1,2\}$ (see [3, Sections 1.3 and 3.2]). If $k = n - 1$ then $G$ is complete, $n = 4$ and $\mu = -1$; otherwise let $d$ be the mean of the eigenvalues other than 3 and $\mu$, so that $3 + k\mu + (n - k - 1)d = 0$. We have $-3 \leq d < 3$; moreover, if $d = -3$ then $G$ is bipartite, $k = n - 2$ and $\mu = 0$ (see [3, Theorems 3.2.3 and 3.2.4]). We deduce:

(a) if $\mu = -2$ then $k \leq \frac{3}{2}n$, i.e. $k \leq \frac{3}{2}t$;
(b) if $\mu = -1$ then $k \leq \frac{1}{2}n$, i.e. $k \leq 3t$;
(c) if $\mu = 0$ then $k \leq n - 2$;
(d) if $\mu = 1$ then $k \leq \frac{3}{2}n - \frac{3}{2}$, i.e. $k < 3t - 6$;
(e) if $\mu = 2$ then $k \leq \frac{3}{2}n - \frac{5}{2}$, i.e. $k < \frac{3}{2}t - 3$.

We use star complements to improve these bounds, and to determine all the graphs for which the new bounds are attained. Our main result is the following; here and throughout we use the notation of the monograph [3].

**Theorem 1.1.** Let $G$ be a connected cubic graph of order $n$ with $\mu$ as an eigenvalue of multiplicity $k$.

(i) If $\mu \not\in \{-1,0\}$ then $k \leq \frac{1}{2}n$, with equality if and only if $\mu = 1$ and $G$ is the Petersen graph.

(ii) If $\mu = -1$ then $k \leq \frac{1}{2}n + 1$, with equality if and only if $G = K_4$.

(iii) If $\mu = 0$ then $k \leq \frac{1}{2}n + 1$, with equality if and only if $G = \overline{2K_3}$.

It follows that if $G$ is a connected cubic graph of order $n > 10$ with $\mu$ as an eigenvalue of multiplicity $k$ then $k \leq \frac{1}{2}n - 1$ when $\mu \not\in \{-1,0\}$, and $k \leq \frac{1}{2}n$ otherwise.

2 Preliminaries

Let $G$ be a graph of order $n$ with $\mu$ as an eigenvalue of multiplicity $k$. A **star set** for $\mu$ in $G$ is a subset $X$ of the vertex-set $V(G)$ such that $|X| = k$ and the induced subgraph $G - X$ does not have $\mu$ as an eigenvalue. In this situation, $G - X$ is called a **star complement** for $\mu$ in $G$. The fundamental properties of star sets and star complements are established in [3, Chapter 5]. We shall require the following results, where for any $X \subseteq V(G)$, we write $G_X$ for the subgraph of $G$ induced by $X$. We take $V(G) = \{1, \ldots, n\}$, and write $u \sim v$ to mean that vertices $u$ and $v$ are adjacent.
Theorem 2.1. (See [3, Theorem 5.1.7].) Let $X$ be a set of $k$ vertices in $G$ and suppose that $G$ has adjacency matrix \( \begin{pmatrix} A_X & B^\top \\ B & C \end{pmatrix} \), where $A_X$ is the adjacency matrix of $G_X$.

(i) Then $X$ is a star set for $\mu$ in $G$ if and only if $\mu$ is not an eigenvalue of $C$ and
\[
\mu I - A_X = B^\top (\mu I - C)^{-1} B.
\]

(ii) If $X$ is a star set for $\mu$ then $E(\mu)$ consists of the vectors $(x(\mu I - C)^{-1} Bx)$ ($x \in \mathbb{R}^k$).

Let $H = G - X$, where $X$ is a star set for $\mu$. The columns $b_u$ ($u \in X$) of $B$ are the characteristic vectors of the $H$-neighbourhoods $\Delta_H(u) = \{ v \in V(H) : u \sim v \}$ ($u \in X$). Eq. (1) shows that
\[
b_u^\top (\mu I - C)^{-1} b_v = \begin{cases} 
\mu & \text{if } u = v \\
-1 & \text{if } u \sim v \\
0 & \text{otherwise,}
\end{cases}
\]
and we deduce from Theorem 2.1:

**Lemma 2.2.** If $X$ is a star set for $\mu$, and $\mu \notin \{-1, 0\}$, then the neighbourhoods $\Delta_H(u)$ ($u \in X$) are non-empty and distinct.

Let $P$ be the matrix of the orthogonal projection of $\mathbb{R}^n$ onto $E(\mu)$ with respect to the standard orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of $\mathbb{R}^n$. Since $P$ is a polynomial in $A$ [3, Equation 1.5] we have $\mu P e_i = A P e_i = P A e_i$ ($i = 1, \ldots, n$), whence:

**Lemma 2.3.** $\mu P e_i = \sum_{j \sim i} P e_j$ ($i = 1, \ldots, n$).

The next observation follows from [3, Proposition 5.1.1].

**Lemma 2.4.** The subset $S$ of $V(G)$ lies in a star set for $\mu$ if and only if the vectors $P e_i$ ($i \in S$) are linearly independent.

By interlacing [3, Corollary 1.3.12] we have:

**Lemma 2.5.** If $S$ is a star set for $\mu$ in $G$ and if $U$ is a proper subset of $S$ then $S \setminus U$ is a star set for $\mu$ in $G - U$.

We shall also require:

**Lemma 2.6.** (See [3, Theorem 5.1.6].) Let $\mu$ be an eigenvalue of the graph $G$. If $G$ is connected then $G$ has a connected star complement for $\mu$.

In the case of connected cubic graphs, we can therefore make use of the following result.

**Proposition 2.7.** Let $G$ be a connected cubic graph of order $n$ with $\mu$ as an eigenvalue of multiplicity $k \geq \frac{1}{2}n$. Let $H$ be a connected star complement for $\mu$, and let $H = G - X$, $\overline{X} = V(H)$, $|\overline{X}| = t$. Then each vertex in $X$ is adjacent to some vertex in $\overline{X}$, and one of the following holds:
(a) \(k = t, |E(X, \overline{X})| = t\) and \(H\) is unicyclic,
(b) \(k = t, |E(X, \overline{X})| = t + 2\) and \(H\) is a tree,
(c) \(k = t + 2, |E(X, \overline{X})| = t + 2, \mu \in \{-1, 0\}\) and \(H\) is a tree.

**Proof.** If \(u \in X\) then \(\mu Pe_u = \sum_{i \in \Delta_X(u)} Pe_i + \sum_{j \in \Delta_H(u)} Pe_j\), where \(\Delta_X(u) = \{i \in X : i \sim u\}\). It now follows from Lemma 2.4 that \(\Delta_H(u) \neq \emptyset\). For \(j \in \overline{X}\), let \(d_j = |\Delta_H(j)|, e_j = |\Delta_X(j)|\).

\[|E(X, \overline{X})| = \sum_{j \in \overline{X}} e_j = 3t - \sum_{j \in X} d_j = 3t - 2|E(H)|.\]

Since \(|E(H)| \geq t - 1\) we deduce that \(|E(X, \overline{X})| \leq t + 2\). Since \(k \geq \frac{1}{2}n\) and each vertex in \(X\) has a neighbour in \(\overline{X}\), we have
\[t \leq k \leq |E(X, \overline{X})| \leq t + 2\quad \text{and} \quad |E(H)| \leq t.\]

If \(|E(H)| = t\) then \(H\) is unicyclic and \(k = |E(X, \overline{X})|\): this is case (a) of the Proposition. If \(|E(H)| = t - 1\) then \(H\) is a tree and \(|E(X, \overline{X})| = t + 2\); moreover, \(k\) is \(t\) or \(t + 2\) because \(n\) is even. If \(k = t\) we have case (b). If \(k = t + 2\) then \(|\Delta_H(i)| = 1\) for each \(i \in X\) and so there are two vertices in \(X\) with a common \(H\)-neighbourhood. We deduce from Lemma 2.2 that \(\mu \in \{-1, 0\}\) and so we have case (c).

It follows that \(k \leq \frac{1}{2}n\) when \(\mu \notin \{-1, 0\}\), and \(k \leq \frac{1}{4}n + 1\) when \(\mu \in \{-1, 0\}\). In Sections 3 and 4 we determine the graphs in which these bounds are attained. It is clear from Proposition 2.7 that the edges between \(X\) and \(\overline{X}\) play a crucial role. The authors of [2] have determined all the graphs for which \(E(X, \overline{X})\) is a perfect matching, equivalently all the graphs for which \(B = I\) in Eq.(1). Their result is the following.

**Theorem 2.8.** Let \(G\) be a graph with \(X\) as a star set for the eigenvalue \(\mu\). If \(E(X, \overline{X})\) is a perfect matching then one of the following holds: (a) \(G = K_2\) and \(\mu = \pm 1\), (b) \(G = C_4\) and \(\mu = 0\), (c) \(G\) is the Petersen graph and \(\mu = 1\).

We shall see that when \(E(X, \overline{X})\) is not a perfect matching, and \(G\) is a connected cubic graph with \(k \geq \frac{1}{2}n\), it suffices to consider a limited number of configurations from which we can construct a fragment of \(G\). In most cases, we invoke Lemmas 2.3 and 2.4 to obtain a contradiction. In the remaining cases, either the fragment is \(G\) itself or we derive a contradiction from Theorem 2.1(ii). The configurations that we consider when \(\mu \notin \{-1, 0\}\) are illustrated in Fig. 1, labelled in accordance with various subcases described in Section 3.

3 The case \(\mu \notin \{-1, 0\}\)

We retain the notation of Section 2. We assume that \(G\) is a connected cubic graph, with \(\mu \notin \{-1, 0\}\) and \(k = \frac{1}{2}n\). Thus \(\mu \in \{-2, 1\}\). By Lemma 2.6, we know that \(G\) has a connected star complement \(H\) for \(\mu\); accordingly we have to deal with cases (a) and (b) of Proposition 2.7. In case (a), the \(t\) edges in \(E(X, \overline{X})\) form a perfect matching (and \(H\) is a cycle) because the vertices in \(X\) have distinct \(H\)-neighbourhoods. Thus \(\mu = 1\) and \(G\) is the Petersen graph, by Theorem 2.8. For the remainder of this section, we therefore assume that \(|E(X, \overline{X})| = t + 2\) and \(H\) is a tree.
that is,

$$\mu^2 Pe_1 = Pe_1 + (\mu + 1)(\mu Pe_2 - v_2) + \mu(\mu Pe_3 - v_3) + \mu Pe_4 - v_4. \quad (2)$$
Now a parity check shows that $\mu = 1$. (If $\mu = \pm 2$ then Eq.(2) can be written in the form $\sum_{i \in X} a_i Pe_i = 0$ with $\sum_{i \in X} a_i \not\equiv 0 \pmod{2}$.) Hence

$$2v_2 + v_3 + v_4 = 2Pe_2 + Pe_3 + Pe_4,$$

and this too contradicts Lemma 2.4.

In subcase (1,2,2), again $\mu Pe_1 = Pe_{1'} + Pe_{2'} + Pe_{2'},$ and now

$$\mu^2 Pe_1 = Pe_1 + Pe_{1'} + Pe_{2'} + \mu Pe_{2'} + \mu Pe_{2'},$$

that is,

$$\mu^2 Pe_1 = Pe_1 + \mu Pe_4 - v_4 + \mu Pe_5 - v_5 + \mu(\mu Pe_2 - v_2) + \mu(\mu Pe_3 - v_3).$$

A parity check shows that $\mu = 1$. Hence

$$v_2 + v_3 + v_4 + v_5 = Pe_2 + Pe_3 + Pe_4 + Pe_5,$$

and this contradicts Lemma 2.4.

It remains to consider case (2), where without loss of generality we take $|\Delta_H(1)| = |\Delta_H(2)| = 2$ and $\Delta_H(i) = \{i'\} (i = 3, \ldots, t)$.

**Lemma 3.1** In Case (2), neither vertex 1 nor vertex 2 is adjacent to two vertices in $\{3', 4', \ldots, t'\}$.

**Proof.** It suffices to rule out the case that $\Delta_H(2) = \{3', 4'\}$. Here we have $\mu Pe_2 = v_2 + Pe_{3'} + Pe_{4'} = v_2 + \mu Pe_3 - v_3 + \mu Pe_4 - v_4$. A parity check shows that $\mu = 1$. Hence

$$Pe_2 + v_3 + v_4 = v_2 + Pe_3 + Pe_4.$$

and this contradicts Lemma 2.4. \[\square\]

In view of Lemma 3.1, we may assume that $\Delta_H(2) = \{2', 3'\}$. We distinguish two subcases: (2,1) $1 \not\sim 1'$, (2,2) $1 \sim 1'$. In subcase (2,1), we have $1 \sim 2'$ by Lemma 3.1. Moreover, since vertices 1 and 2 have distinct $H$-neighbourhoods, we may assume that $\Delta_H(1) = \{2', 4'\}$. Now we have

$$\mu Pe_1 = v_1 + Pe_{2'} + Pe_{3'} = v_1 + \mu Pe_2 - Pe_3 - v_2 + \mu Pe_4 - v_4$$

$$= v_1 + \mu Pe_2 - \mu Pe_3 + v_3 - v_2 + \mu Pe_4 - v_4.$$

If $\mu = 2$ then

$$2Pe_1 + 2Pe_3 + v_2 + v_4 = 2Pe_2 + 2Pe_4 + v_1 + v_3,$$

and we obtain a contradiction by equating coefficients of $Pe_1$.

If $\mu = -2$ then

$$2Pe_1 + 2Pe_3 + v_1 + v_3 = 2Pe_2 + 2Pe_4 + v_2 + v_4,$$

whence $v_2 = Pe_1 + Pe_3$, a contradiction.

Hence $\mu = 1$ and we have

$$Pe_1 + Pe_3 + v_2 + v_4 = Pe_2 + Pe_4 + v_1 + v_3.$$
For both values of \( \mu \), \( \Delta_X(1) = \{3\} \), \( \Delta_X(2) = \{4\} \), \( \Delta_X(3) = \{1, h\} \) and \( \Delta_X(4) = \{2, h\} \) for some \( h > 4 \). Without loss of generality, \( h = 5 \). Thus the vertices 1, 2, 3, 4, 5 induce a path which is component of \( G_X \), while any other component of \( G_X \) is a cycle.

By Theorem 2.1(ii), \( G \) has a 1-eigenvector \( x = (x(i))_{i \in V(G)} \) such that \( x(1) = 1 \) and \( x(i) = 0 \) (\( i = 2, \ldots, t \)). By Lemma 2.3, we have \( x(i') = 0 \) for all \( i \geq 5 \). Let \( x(2') = a \), so that \( x(3') = -a \) and \( x(4') = 1 - a \). For \( i = 2, 3, 4 \), let \( \Delta_H(i') = \{i''\} \). Then \( x(2'') = a - 1 \), \( x(3'') = 0 \) and \( x(4'') = -a \). Since vertices 2', 3', 4' are endvertices of \( H \), they constitute an independent set. Thus if \( 3' \sim 1' \) then \( x(1') = 0 \) and so \( x(2'') = x(4'') = 0 \), a contradiction. Hence \( 3' \sim j' \) for some \( j \geq 5 \) and we have:

\[
P e_2 = P e_{2'} + P e_{3'} + P e_4 = P e_1 - P e_{4'} - P e_3 + P e_2 + P e_3 + P e_{j'} + P e_4
\]

\[
= P e_i - P e_4 + v_1 - P e_3 + P e_2 + P e_3 + P e_j - v_j + P e_4.
\]

Hence \( v_j = P e_1 + P e_j + v_4 \), a contradiction.

Now we turn to subcase (2,2), where \( 1' \sim 1 \neq 3' \) and we may assume that either (2,2,1) \( 1 \sim 2' \) or (2,2,2) \( 1 \sim 4' \). In subcase (2,2,1), \( H \) has degree sequence \( (1, 2, (t - 2)) \), and so \( H \) is a path; its endvertices are \( 2' \) and \( 3' \). Since \( \Delta_H(2) = \{2', 3'\} \), the subgraph of \( G \) induced by \( V(H) \cup \{2\} \) is a \((t + 1)\)-cycle. By Lemma 2.5, this subgraph has \( \mu \) as a simple eigenvalue, and so \( \mu = \pm 2 \).

Since \( 1' \) is not adjacent to both \( 2' \) and \( 3' \), we should consider just three possibilities: (2,2,1,1) \( \Delta_H(1') = \{4', 5'\} \), (2,2,1,2) \( \Delta_H(1') = \{2', 4'\} \), (2,2,1,3) \( \Delta_H(1') = \{3', 4'\} \).

In subcase (2,2,1,1) we have \( \mu P e_1 = v_1 + P e_{4'} + P e_{2'} \), whence

\[
\mu^2 P e_1 = \mu v_1 + P e_1 + P e_{4'} + P e_{2'} + \mu P e_{2'}
\]

\[
= \mu v_1 + P e_1 + \mu P e_4 - v_4 + \mu P e_5 - v_5 + \mu(\mu P e_2 - v_2 - P e_{4'})
\]

\[
= \mu v_1 + P e_1 + \mu P e_4 - v_4 + \mu P e_5 - v_5 + \mu^2 P e_2 - \mu v_2 - \mu(\mu P e_3 - v_3).
\]

Now a parity check gives a contradiction.

In subcase (2,2,1,2), we have \( \mu P e_1 = v_1 + P e_{1'} + P e_{2'} \), and so

\[
\mu^2 P e_1 = \mu v_1 + P e_1 + P e_{4'} + P e_{2'} + \mu P e_{2'} = \mu v_1 + P e_1 + \mu P e_4 - v_4 + (\mu + 1) P e_{2'}
\]

\[
= \mu v_1 + P e_1 + \mu P e_4 - v_4 + (\mu + 1)(\mu P e_2 - v_2 - P e_{4'})
\]

\[
= \mu v_1 + P e_1 + \mu P e_4 - v_4 + (\mu + 1)(\mu P e_2 - v_2 - \mu P e_3 + v_3).
\]

If \( \mu = 2 \) then

\[
3 P e_1 + v_4 + 3 v_2 + 6 P e_3 = 2 v_1 + 2 P e_4 + 6 P e_2 + 3 v_3.
\]

If \( \mu = -2 \) then

\[
3 P e_1 + 2 v_1 + 2 P e_4 + v_4 + 2 P e_3 + v_3 = 2 P e_2 + v_2.
\]

For both values of \( \mu \), Lemma 2.4 is contradicted.
In subcase (2,2,1,3), we have $\mu Pe_1 = v_1 + Pe_1 + Pe_2$ and so
\[
\mu^2 P e_1 = \mu v_1 + Pe_1 + Pe_2 + Pe_3 + \mu Pe_2
\]
\[
= \mu v_1 + Pe_1 + \mu Pe_3 - v_3 + \mu Pe_4 - v_4 + \mu Pe_2
\]
\[
= \mu v_1 + Pe_1 + \mu Pe_3 - v_3 + \mu Pe_4 - v_4 + \mu(\mu Pe_2 - v_2 - \mu Pe_3 + v_3).
\]
Again a parity check gives a contradiction.

Now we consider subcase (2,2,2), where $1 \sim 4'$ and $H$ is a path with end-vertices $3'$ and $4'$. By Lemma 2.5 the subgraph of $G$ induced by $V(H) \cup \{3, 4\}$ has $\mu$ as a double eigenvalue; hence this subgraph is a $(t+2)$-cycle, and $\mu = 1$. Let $\Delta_H(3') = \{i\}'$, and let $H_i$ be the subgraph induced by $V(H) \cup \{i\}$. Then $i \in \{1, 2\}$ for otherwise $H_i$ is a tree without a 1-eigenvector $x$ such that $x(i) = 1$. Similarly, $\Delta_H(4') = \{j'\}$, where $j \in \{1, 2\}$. Since $t > 3$ we have $i \neq j$, and so either (2,2,1) $\Delta_X(3') = \{2'\}$, $\Delta_X(4') = \{1'\}$ or (2,2,2) $\Delta_X(3') = \{1'\}$, $\Delta_X(4') = \{2'\}$.

In subcase (2,2,2,1), we have $\mu Pe_4 = Pe_4 + v_4$, whence
\[
\mu^2 Pe_4 = Pe_4 + Pe_1 + Pe_2 + v_4 = Pe_4 + Pe_1 + Pe_2 - v_1 + v_4
\]
Since $\mu = 1$, we have
\[
Pe_4 + v_1 = 2Pe_1 + 2v_4,
\]
contradicting Lemma 2.4.

In subcase (2,2,2,2), we have $\mu Pe_4 = Pe_4 + v_4$ and
\[
\mu^2 Pe_4 = Pe_4 + Pe_1 + Pe_2 + v_4 = Pe_4 + Pe_1 + Pe_2 - v_2 + v_4
\]
Since $\mu = 1$, we have
\[
Pe_3 + v_2 = Pe_1 + Pe_2 + v_3 + v_4,
\]
contradicting Lemma 2.4.

We have now proved:

**Proposition 3.2.** Let $G$ be a connected cubic graph of order $n$ with an eigenvalue $\mu$ of multiplicity $\frac{1}{2}n$. If $\mu \not\in \{-1, 0\}$ then $\mu = 1$, $n = 10$ and $G$ is the Petersen graph.

4 The case $\mu \in \{-1, 0\}$

In this section we assume that $G$ is a connected cubic graph, with $\mu \in \{-1, 0\}$ and $k = \frac{1}{2}n + 1$ (that is, $k = t + 2$). By Lemma 2.6, we know that $G$ has a connected star complement for $\mu$, say $H = G - X$. By Proposition 2.7, $H$ is a tree; moreover $|\Delta_H(u)| = 1$ for all $u \in X$, and so $G_X$ is a union of disjoint cycles. Note that there exist (at least) two vertices in $X$ with a common neighbour in $H$. 

7
Lemma 4.1. Let $G$ be a graph with $X$ as a star set for the eigenvalue $\mu$, and let $H = G - X$. Suppose that $u, v$ are distinct vertices in $X$ such that $\Delta_H(u) = \Delta_H(v)$.

(i) If $\mu = -1$ then $\Delta_X(u) \cup \{u\} = \Delta_X(v) \cup \{v\}$ (and so $u, v$ are co-duplicate vertices).

(ii) If $\mu = 0$ then $\Delta_X(u) = \Delta_X(v)$ (and so $u, v$ are duplicate vertices).

Proof. Both (i) and (ii) follow from Lemma 2.4 and the relation

$$\mu Pe_u - \sum_{i \in \Delta_X(u)} Pe_i = \mu Pe_v - \sum_{j \in \Delta_X(v)} Pe_j.$$ 

□

Let $X = \{1, 2, \ldots, t + 2\}$, $\overline{X} = \{1', 2', \ldots, t'\}$, with $\Delta_H(1) = \Delta_H(2) = \{1\}$. Suppose first that $\mu = -1$. By Lemma 4.1(i), we have $1 \sim 2$, and we may take $\Delta_X(1) = \{2, 3\}$, $\Delta_X(2) = \{1, 3\}$. This argument shows that no vertex of $H$ is adjacent to two vertices in different components of $G_X$.

If $3 \sim 1'$ then $G = K_4$, and so we suppose that $3 \sim 2'$. By Theorem 2.1(ii), $G$ has a $(-1)$-eigenvector $x$ with $x(1) = 1$ and $x(i) = 0$ ($i = 2, 3, \ldots, t + 2$). We have $x(1') = x(2') = -1$. Consider an $r$-cycle $C$ other than 1231 in $G_X$. If $C$ has two vertices with a common neighbour in $H$ then $r = 3$, and by Lemma 2.3, $x(i') = 0$ for each neighbour $i'$ in $H$ of a vertex of $C$. The same conclusion holds when $C$ does not have two vertices with a common neighbour in $H$. It follows that $x(i') = 0$ ($i = 3, \ldots, t$). Thus the non-zero entries of $x$ are 1, -1, -1, and $x$ is not orthogonal to the all-1 vector $j \in \mathbb{R}^n$. This is a contradiction because $j$ is a 3-eigenvector of $G$.

Next suppose that $\mu = 0$. By Lemma 4.1(ii), we may take $\Delta_X(1) = \Delta_X(2) = \{3, 4\}$, where $3 \not\sim 1' \not\sim 4$; moreover, $3 \not\sim 4$ because $\Delta_H(4) \neq \emptyset$. Note that again no vertex of $H$ is adjacent to two vertices in different components of $G_X$. Now let $x$ be a 0-eigenvector with $x(1) = 1$ and $x(i) = 0$ ($i = 2, \ldots, t + 2$). Note that $x(1') = 0$, and consider an $r$-cycle $C$ other than 13241 in $G_X$. If $C$ has two vertices with a common neighbour in $H$ then $r = 4$, and by Lemma 2.3, $x(i') = 0$ for each neighbour $i'$ in $H$ of a vertex in $C$. The same conclusion holds when $C$ does not have two vertices with a common neighbour in $H$.

If vertices 3 and 4 have a common neighbour in $H$, say $2'$, then $x(2') = -1$; moreover if $\Delta_H(1') = \{j'\}$ then $x(j') = -1$, while $x(i') = 0$ ($i = 3, \ldots, t$). In this case, $j = 2$ and $G = \overline{2K_3}$. If vertices 3 and 4 have different neighbours in $H$, say $\Delta_H(3) = \{2'\}$ and $\Delta_H(4) = \{3'\}$ then $x(2') = x(3') = -1$, while $x(i') = 0$ ($i = 4, \ldots, t$). Now $j^+x \neq 0$, a contradiction as before. We have therefore proved:

Proposition 4.2. Let $G$ be a connected cubic graph of order $n$ with an eigenvalue $\mu$ of multiplicity $\frac{1}{2} n + 1$. If $\mu = -1$ then $G = K_4$, and if $\mu = 0$ then $G = \overline{2K_3}$.

In view of Lemma 2.6, we can combine Propositions 2.7, 3.2 and 4.2 to obtain Theorem 1.1.
References


